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ASYMPTOTIC DISTRIBUTION OF NODAL INTERSECTIONS FOR ARITHMETIC RANDOM WAVES

MAURIZIA ROSSI AND IGOR WIGMAN

ABSTRACT. We study the nodal intersections number of random Gaussian toral Laplace eigenfunctions (“arithmetic random waves”) against a fixed smooth reference curve. The expected intersection number is proportional to the the square root of the eigenvalue times the length of curve, independent of its geometry. The asymptotic behaviour of the variance was addressed by Rudnick-Wigman; they found a precise asymptotic law for “generic” curves with nowhere vanishing curvature, depending on both its geometry and the angular distribution of lattice points lying on circles corresponding to the Laplace eigenvalue. They also discovered that there exist peculiar “static” curves, with variance of smaller order of magnitude, though did not prescribe what the true asymptotic behaviour is in this case.

In this paper we study the finer aspects of the limit distribution of the nodal intersections number. For “generic” curves we prove the Central Limit Theorem (at least, for “most” of the energies). For the aforementioned static curves we establish a non-Gaussian limit theorem for the distribution of nodal intersections, and on the way find the true asymptotic behaviour of their fluctuations, under the well-separatedness assumption on the corresponding lattice points, satisfied by most of the eigenvalues.

KEYWORDS AND PHRASES: ARITHMETIC RANDOM WAVES, NODAL INTERSECTIONS, LIMIT THEOREMS, LATTICE POINTS.

AMS CLASSIFICATION: 60G60, 60B10, 60D05, 35P20, 58J50

1. INTRODUCTION AND MAIN RESULTS

1.1. Toral nodal intersections. Let $\mathbb{T} = \mathbb{R}^2/\mathbb{Z}^2$ be the two-dimensional standard torus and Δ the Laplacian on \mathbb{T} . It is well-known that the eigenvalues of $-\Delta$ (“energy levels”) are all the number of the form $E_n = 4\pi^2 n$ where n is an integer expressible as a sum of two squares

$$n \in S := \{a^2 + b^2 : a, b \in \mathbb{Z}\}.$$

Given a number $n \in S$ we denote

$$\Lambda_n = \{\lambda = (\lambda_1, \lambda_2) \in \mathbb{Z}^2 : \|\lambda\|^2 := \lambda_1^2 + \lambda_2^2 = n\}$$

to be the collection of lattice points lying on the radius- \sqrt{n} centred circle in \mathbb{R}^2 , and $\mathcal{N}_n = |\Lambda_n|$ to be their number; the eigenspace of $-\Delta$ corresponding to E_n then admits the orthonormal basis

$$\{e_\lambda(x) := e^{i2\pi\langle\lambda, x\rangle}\}_{\lambda \in \Lambda_n},$$

$x = (x_1, x_2) \in \mathbb{T}$. Equivalently, we may express every (complex-valued) function T_n satisfying

$$\Delta T_n + E_n T_n = 0$$

as a linear combination

$$(1.1) \quad T_n(x) := \frac{1}{\sqrt{\mathcal{N}_n}} \sum_{\lambda \in \Lambda_n} a_\lambda e_\lambda(x)$$

(the meaning of the normalizing constant on the r.h.s. of (1.1) will be clear in §1.2); T_n is real-valued if and only if for every $\lambda \in \Lambda_n$ we have

$$(1.2) \quad a_{-\lambda} = \overline{a_\lambda}.$$

From now on, we assume T_n in (1.1) to be real-valued. The *nodal line* of T_n is the zero set $T_n^{-1}(0)$; under some generic assumptions $T_n^{-1}(0)$ is a *smooth curve*. Given a fixed reference curve $\mathcal{C} \subset \mathbb{T}$ one is interested in the number $\mathcal{Z}_{\mathcal{C}}(T_n)$ of *nodal intersection*, i.e. the number of intersections of $T_n^{-1}(0)$ with \mathcal{C} ; one expects that for n sufficiently big $\mathcal{Z}_{\mathcal{C}}(T_n)$ is *finite*, and it is believed that their number should be commensurable with \sqrt{n} . That it is indeed so was verified by Bourgain and Rudnick [BR12] for \mathcal{C} with nowhere vanishing curvature; they showed that in this case

$$(1.3) \quad n^{1/2-o(1)} \ll \mathcal{Z}_{\mathcal{C}}(T_n) \ll \sqrt{n},$$

and got rid [BR15] of the $o(1)$ in the exponent of the lower bound for “most” n .

1.2. Nodal intersections for arithmetic random waves. We endow the linear space (1.1) with a Gaussian probability measure by taking the coefficients a_λ in (1.1) random variables. Namely, we assume that the a_λ are standard complex-Gaussian i.i.d. save to (1.2), all defined on the same probability space; equivalently, T_n is a real-valued centered Gaussian field on \mathbb{T} with covariance function

$$(1.4) \quad r_n(x, y) := \mathbb{E}[T_n(x) \cdot T_n(y)] = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi \langle \lambda, x - y \rangle).$$

The random fields T_n are the “arithmetic random waves” [ORW08, KKW13, MPRW16]; by (1.4) above, T_n are unit variance and stationary, and by the standard abuse of notation we may denote

$$r_n(x - y) := r_n(x, y).$$

Given a smooth (finite) length- L curve $\mathcal{C} \subset \mathbb{T}^2$ we define $\mathcal{Z}_n = \mathcal{Z}_{\mathcal{C}}(T_n)$ to be the number of nodal intersections of T_n against \mathcal{C} ; it is a (a.s. finite [RW16]) random variable whose distribution is the primary focus of this paper. Rudnick and Wigman [RW16] have computed its expected number to be

$$(1.5) \quad \mathbb{E}[\mathcal{Z}_n] = \frac{\sqrt{E_n}}{\pi\sqrt{2}} L$$

independent of the geometry of \mathcal{C} , and also studied the asymptotic behaviour of the variance of \mathcal{Z}_n for large values of n ; in order to be able to exhibit their results we require some number theoretic preliminaries. First, denote $\mathcal{N}_n = |\Lambda_n|$ to be the number of lattice points lying on the radius- \sqrt{n} circle. While on one hand, along $n \in S$ the number \mathcal{N}_n grows [Lan08] *on average* as $c_{RL} \cdot \sqrt{\log n}$ with $c_{RL} > 0$ the Ramanujan-Landau constant, on the other hand \mathcal{N}_n is subject to large and erratic fluctuations; for example for the (thin) sequence of primes $p \equiv 1 \pmod{4}$ the corresponding $\mathcal{N}_p = 8$ does not grow at all. From this point on we will assume that

$$\mathcal{N}_n \rightarrow \infty$$

(also holding for a density-1 sequence $\{n\} \subset S$); it is also easy to derive [MV07, p. 130] the bound

$$(1.6) \quad \mathcal{N}_n = O(n^{o(1)}).$$

We will also need to consider the angular distribution of Λ_n ; to this end we define the probability measures

$$(1.7) \quad \mu_n := \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \delta_{\lambda/\sqrt{n}},$$

$n \in S$, on the unit circle $\mathcal{S}^1 \subset \mathbb{R}^2$. It is well-known that for a density-1 subsequence $\{n_j\} \subset S$ the corresponding lattice points Λ_{n_j} are asymptotically equidistributed in the sense that the corresponding measures are weak-* convergent to the uniform measure on \mathcal{S}^1 , usually denoted

$$(1.8) \quad \mu_{n_j} \Rightarrow \frac{d\theta}{2\pi}.$$

To the other extreme, there exists [Cil93] a (thin) sequence $\{n_j\} \subset S$ with angles all concentrated around $\pm 1, \pm i$

$$(1.9) \quad \mu_{n_j} \Rightarrow \tau_0 := \frac{1}{4} (\delta_{\pm 1} + \delta_{\pm i})$$

(thinking of $\mathcal{S}^1 \subset \mathbb{C}$), and the other partial weak-* limits of $\{\mu_n\}_{n \in S}$ were partially classified [KKW13, KW17] (“attainable measures”). The measure τ_0 (“Cilleruelo measure”) together with its rotation by $\frac{\pi}{4}$ (“tilted Cilleruelo”) are the two unique measures supported on 4 points only, that are invariant w.r.t. rotation by $\frac{\pi}{2}$; both τ_0 and its $\frac{\pi}{4}$ -tilt are weak-* partial limits of $\{\mu_n\}_{n \in S}$.

Back to the variance of \mathcal{Z}_n , let

$$\gamma = (\gamma_1, \gamma_2) : [0, L] \rightarrow \mathbb{T}$$

be the arc-length parametrization of \mathcal{C} . Rudnick and Wigman [RW16, Theorem 1.1] found that¹

$$(1.10) \quad \text{Var}(\mathcal{Z}_n) = (4B_{\mathcal{C}}(\Lambda_n) - L^2) \cdot \frac{n}{\mathcal{N}_n} + O\left(\frac{n}{\mathcal{N}_n^{3/2}}\right),$$

where

$$(1.11) \quad \begin{aligned} B_{\mathcal{C}}(\Lambda_n) &:= \int_0^L \int_0^L \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_1) \right\rangle^2 \cdot \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_2) \right\rangle^2 dt_1 dt_2 \\ &= \int_0^L \int_0^L \int_{\mathcal{S}^1} \langle \theta, \dot{\gamma}(t_1) \rangle^2 \cdot \langle \theta, \dot{\gamma}(t_2) \rangle^2 d\mu_n(\theta) dt_1 dt_2. \end{aligned}$$

The leading term

$$4B_{\mathcal{C}}(\Lambda_n) - L^2$$

fluctuates [RW16, §7] in the interval $[0, L^2]$ depending on both the angular distribution of the lattice points Λ_n and the geometry of \mathcal{C} . One may also define $B_{\mathcal{C}}(\mu)$ for any probability measure μ on \mathcal{S}^1 , invariant w.r.t. rotation by $\frac{\pi}{2}$, with μ in place of μ_n on the r.h.s. of (1.11).

¹Initially under another technical assumption, subsequently lifted in [RWY16].

In general, $B_{\mathcal{C}}(\mu)$ depends both on the geometry of γ , and μ . One might invert the integration order in (1.11) to write

$$(1.12) \quad B_{\mathcal{C}}(\mu) = \int_{\mathcal{S}^1} E(\gamma; \theta)^2 d\mu,$$

where

$$E(\gamma; \theta) = \int_{\mathcal{C}} \langle \theta, \dot{\gamma}(t) \rangle^2 dt,$$

is the “ L^2 -energy of γ along direction θ ”, satisfying

$$(1.13) \quad E(\gamma; \theta) + E(\gamma; \theta^\perp) = L$$

for θ^\perp the rotation of θ by $\frac{\pi}{2}$. Hence, under the constraint (1.13), for every $\theta \in \mathcal{S}^1$ we have

$$(1.14) \quad \frac{L^2}{2} \leq E(\gamma; \theta)^2 + E(\gamma; \theta^\perp)^2 \leq L^2.$$

Upon substituting (1.14) into (1.12), and using the $\frac{\pi}{2}$ -invariance of μ , we may deduce that

$$\frac{L^2}{4} \leq B_{\mathcal{C}}(\mu) \leq \frac{L^2}{2},$$

so that the leading term in (1.10) satisfies

$$(1.15) \quad 0 \leq 4B_{\mathcal{C}}(\mu) - L^2 \leq L^2.$$

In order for the variance (1.10) to observe an asymptotic law we need to split S into sequences $\{n_j\} \subset S$ with corresponding Λ_{n_j} admitting limit angular law. That is, $\mu_{n_j} \Rightarrow \mu$ for some μ probability measure on \mathcal{S}^1 ; in this case

$$B_{\mathcal{C}}(\Lambda_{n_j}) = B_{\mathcal{C}}(\mu_{n_j}) \rightarrow B_{\mathcal{C}}(\mu),$$

so that if $B_{\mathcal{C}}(\mu) > L^2/4$, in this case (1.10) is

$$\text{Var}(\mathcal{Z}_{n_j}) \sim (4B_{\mathcal{C}}(\mu) - L^2) \frac{n_j}{\mathcal{N}_{n_j}}.$$

Rudnick and Wigman [RW16] observed that there exist “special” curves (see Definition 1 in §1.3) for which

$$B_{\mathcal{C}}(\mu) \equiv \frac{L^2}{4}$$

for all μ probability measures on \mathcal{S}^1 invariant w.r.t. rotation by $\frac{\pi}{2}$ so that the leading term vanishes irrespective of the limit measure (though, given \mathcal{C} the equality

$$B_{\mathcal{C}}(\mu) = \frac{L^2}{4}$$

might still hold for some μ without holding identically, very restrictive in terms of μ); for these (1.10) gives no clue as of what is the true behaviour of $\text{Var}(\mathcal{Z}_n)$ other than that it is of lower order of magnitude than the “typical” n/\mathcal{N}_n .

1.3. Statement of main results: limiting laws for \mathcal{Z}_n . Our two principal results below concern the limit laws for \mathcal{Z}_n . Theorem 1.1 asserts the Central Limit Theorem for \mathcal{Z}_n , under the “generic” scenario, where the leading terms of the variance in (1.10) are bounded away from zero (cf. §1.4 for more details on how generic this situation is). Indeed, if we assume that the lattice points Λ_n are equidistributed (i.e. the generic assumption (1.8) on the energy levels), then the assumptions of Theorem 1.1 hold for generic curves [RW16, Corollary 7.2], also see the discussion in §1.4 below. Theorem 1.3 investigates the peculiar alternative situation (the slightly more restrictive aforementioned “static” curves) with a non-Gaussian limit law, and among other things determines the true (lower order) asymptotic law of the variance (1.10) in the latter case.

Conventions. The notation \rightarrow^d means convergence in distribution of random variables, $=^d$ denotes equality in law between two random variables or random fields, and $\mathcal{N}(m, \sigma^2)$ is the Gaussian distribution with mean m and variance σ^2 . Throughout this manuscript we will assume that $\mathcal{C} \subset \mathbb{T}$ is a given (fixed) curve of length L , and $\gamma : [0, L] \rightarrow \mathbb{T}$ an arc-length parametrization of \mathcal{C} .

First we formulate the Central Limit Theorem holding under “generic” assumptions.

Theorem 1.1. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth curve on the torus with nowhere zero curvature of total length L , and $\{n\} \subset S$ such that $\mathcal{N}_n \rightarrow +\infty$, and $\{4B_{\mathcal{C}}(\mu_n) - L^2\}$ is bounded away from zero. Then the limiting distribution of the nodal intersections number is Gaussian, i.e.*

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} \xrightarrow{d} Z,$$

where $Z \sim \mathcal{N}(0, 1)$.

Next investigate the (non-generic) situation when the variance is of lower order. In order to formulate Theorem 1.3 we will have to restrict \mathcal{C} to be static (Definition 1), and the sequence $\{n\} \subset S$ to be δ -separated (Definition 2).

Definition 1 (Static curves). *A smooth curve $\mathcal{C} \subset \mathbb{T}$ with nowhere zero curvature is called static if for every probability measure μ on \mathcal{S}^1*

$$(1.16) \quad 4B_{\mathcal{C}}(\mu) - L^2 = 0.$$

For example, any semi-circle or circle are static [RW16, §7.2]. In Appendix G we show that any smooth curve with nowhere vanishing curvature and invariant under some nontrivial rotation of finite order, is static. We thank D. Panov for pointing this out.

Definition 2 (δ -separated sequences). *Let $\delta > 0$. A sequence $\{n\} \subset S$ of energy levels is δ -separated if*

$$(1.17) \quad \min_{\lambda \neq \lambda' \in \Lambda_n} \|\lambda - \lambda'\| \gg n^{1/4+\delta}.$$

Bourgain and Rudnick [BR11, Lemma 5] showed that “most” n satisfy the δ -separatedness property for every $0 < \delta < \frac{1}{4}$. In fact they have a strong quantitative estimate on the number of the exceptions; a precise estimate on these was established more recently [GW17].

Now we introduce some more notation.

Notation 1.2. (1) For $t \in [0, L]$ set

$$(1.18) \quad \begin{aligned} f(t) &:= \dot{\gamma}_1(t)^2 - \frac{1}{L} \int_0^L \dot{\gamma}_1(u)^2 du = -\dot{\gamma}_2(t)^2 + \frac{1}{L} \int_0^L \dot{\gamma}_2(u)^2 du, \\ g(t) &:= \dot{\gamma}_1(t)\dot{\gamma}_2(t) - \frac{1}{L} \int_0^L \dot{\gamma}_1(u)\dot{\gamma}_2(u) du. \end{aligned}$$

(2) For a probability measure μ on \mathcal{S}^1 define

$$(1.19) \quad A_C(\mu) := \int_{\mathcal{S}^1} \int_{\mathcal{S}^1} \left(\int_0^L \langle \theta, \dot{\gamma}(t) \rangle^2 \cdot \langle \theta', \dot{\gamma}(t) \rangle^2 dt \right)^2 d\mu(\theta) d\mu(\theta').$$

(3) Also define the random variable

$$(1.20) \quad \mathcal{M}(\mu) := \frac{1}{\sqrt{16A_C(\mu) - L^2}} (a_1(\mu)(Z_1^2 - 1) + a_2(\mu)(Z_2^2 - 1) + a_3(\mu)Z_1Z_2),$$

with Z_1, Z_2 i.i.d. standard Gaussian random variables and

$$(1.21) \quad \begin{aligned} a_1(\mu) &:= 2(1 + \widehat{\mu}(4)) \int_0^L f(t)^2 dt, & a_2(\mu) &:= 2(1 - \widehat{\mu}(4)) \int_0^L f(t)^2 dt, \\ a_3(\mu) &:= 4\sqrt{1 - \widehat{\mu}(4)^2} \int_0^L f(t)g(t) dt, \end{aligned}$$

where

$$\widehat{\mu}(4) := \int_{\mathcal{S}^1} z^{-4} d\mu(z)$$

is the 4th Fourier coefficient of μ ; $\widehat{\mu}(4)$ is the first nontrivial Fourier coefficient, as $\widehat{\mu}(0) = 1$, and, by the $\frac{\pi}{2}$ -invariance of μ , we have $\widehat{\mu}(k) = 0$ unless k is divisible by 4.

Alternatively we can define the quantities in (1.18) in the following way. For every $t \in [0, L]$ let $\phi(t) \in [0, 2\pi]$ be the argument of $\dot{\gamma}(t)$, where $\dot{\gamma}(t)$ is viewed as a (unit modulus) complex number, i.e.

$$(1.22) \quad \dot{\gamma}(t) =: e^{i\phi(t)},$$

then

$$\begin{aligned} f(t) &= \cos^2 \phi(t) - \frac{1}{L} \int_0^L \cos^2 \phi(u) du = -\sin^2 \phi(t) + \frac{1}{L} \int_0^L \sin^2 \phi(u) du, \\ g(t) &= \cos \phi(t) \sin \phi(t) - \frac{1}{L} \int_0^L \cos \phi(u) \sin \phi(u) du. \end{aligned}$$

We are now in a position to formulate our second principal theorem.

Theorem 1.3. Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L , and $\{n\} \subset S$ a δ -separated sequence of energies such that $\mathcal{N}_n \rightarrow +\infty$.

(1) The variance of \mathcal{Z}_n is asymptotic to

$$(1.23) \quad \text{Var}(\mathcal{Z}_n) = \frac{n}{4\mathcal{N}_n^2} (16A_C(\mu_n) - L^2) \cdot (1 + o(1)),$$

with the leading term $16A_C(\mu_n) - L^2$ bounded away from zero under the sole assumption that \mathcal{C} is static.

- (2) *There exists a coupling of the random variables \mathcal{Z}_n and $\mathcal{M}(\mu_n)$ (defined in (1.20)), such that*

$$\mathbb{E} \left[\left| \frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} - \mathcal{M}(\mu_n) \right| \right] \rightarrow 0,$$

and

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} - \mathcal{M}(\mu_n) \rightarrow 0, \quad \text{a.s.}$$

The second part of Theorem 1.3 establishes that the distribution law of the normalised intersections number

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}}$$

is asymptotic to that of $\mathcal{M}(\mu_n)$, both in L^1 -sense and a.s. The quantity $A_{\mathcal{C}}(\mu_n)$ (and also $16A_{\mathcal{C}}(\mu_n) - L^2$) depend on both the geometry of \mathcal{C} and μ_n in a nontrivial way. In the particular case of \mathcal{C} a full circle, and $\mu_n \Rightarrow d\theta/2\pi$, Theorem 1.3 yields the following via a routine computation:

Example 1.4. *Let $\mathcal{C} \subset \mathbb{T}$ be a full circle of total length L , let $\{n\} \subset S$ be a δ -separated sequence such that $\mathcal{N}_n \rightarrow +\infty$ and $\mu_n \Rightarrow \frac{d\theta}{2\pi}$, then for the variance of the nodal intersections number we have*

$$\text{Var}(\mathcal{Z}_n) \sim \frac{L^2}{32} \cdot \frac{n}{\mathcal{N}_n^2},$$

and the limiting distribution is

$$(1.24) \quad \frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} \xrightarrow{d} 1 - \frac{Z_1^2 + Z_2^2}{2},$$

where Z_1, Z_2 are i.i.d. standard Gaussian random variables. Note that the r.h.s. of (1.24) is bounded above by 1; to our best knowledge the l.h.s. of (1.24) might be bigger than 1, so, in light of the above, bounded above by 1 with high probability as $n \rightarrow \infty$.

1.4. Discussion. Given a length- L toral curve \mathcal{C} and its arc-length parametrization

$$\gamma : [0, L] \rightarrow \mathbb{T}$$

we may [RW16, §7.3] associate a complex number $I(\gamma) \in \mathbb{C}$ in the following way. Let us keep in mind (1.22): for every $t \in [0, L]$ let $\phi(t) \in [0, 2\pi]$ be the argument of $\dot{\gamma}(t)$, where $\dot{\gamma}(t)$ is viewed as a (unit modulus) complex number, i.e. $\dot{\gamma}(t) = e^{i\phi(t)}$, we then set

$$I(\gamma) := \int_0^L e^{2i\phi(t)} dt.$$

[RW16, Corollary 7.2] asserts that:

- (1) The curve \mathcal{C} is static if and only if $I(\gamma) = 0$.
- (2) If $I(\gamma) \neq 0$ and $\Re(I(\gamma)) = 0$ (resp. $\Im(I(\gamma)) = 0$), then for all μ measures on \mathcal{S}^1 invariant w.r.t. rotation by $\frac{\pi}{2}$ and complex conjugation, $4B_{\mathcal{C}}(\mu) - L^2 = 0$ if and only if $\mu = \tau_0$ is the Cilleruelo measure on the r.h.s. of (1.9) (resp. $4B_{\mathcal{C}}(\mu) - L^2 = 0$ if and only if μ is the $\frac{\pi}{4}$ -tilt of τ_0 on the r.h.s. of (1.9)).

It follows from above that if both the imaginary and real parts

$$(1.25) \quad \Im(I(\gamma)), \Re(I(\gamma)) \neq 0$$

of $I(\gamma)$ do not vanish, then [RW16, Corollary 7.2] show that $4B_{\mathcal{C}}(\mu) - L^2$ is bounded away from zero for *all* probability measures μ on \mathcal{S}^1 invariant w.r.t. rotation by $\frac{\pi}{2}$ and complex conjugation. Hence in this case (1.25), the assumptions of Theorem 1.1 are satisfied for the full sequence $n \in S$ of energy levels (still assuming $\mathcal{N}_n \rightarrow +\infty$). The condition (1.25) is a generic condition on \mathcal{C} , understood, for example, in the sense of *prevalence*, see² e.g. [OY05, §6] (in particular Example 3.6) and references therein. Hence the scenario described by Theorem 1.1 is “generic”: it holds for both “generic” curves for all energy levels, and also the non-generic curves with $I(\gamma) \in \mathbb{R}, i\mathbb{R}$ either real or purely imaginary, for generic energy levels. Note that the condition 1.25 is *not* invariant under rotations of γ , so neither is the distribution of \mathcal{Z}_n , nor its variance.

The two only cases not covered by theorems 1.1 and 1.3 are then:

- (1) We have $\Re(I(\gamma)) = 0$, $\Im(I(\gamma)) \neq 0$ and the lattice points corresponding to the subsequence $\{n\} \subseteq S$ converge to

$$\mu_n \Rightarrow \frac{1}{4}(\delta_{\pm 1} + \delta_{\pm i})$$

the Cilleruelo measure.

- (2) We have $\Re(I(\gamma)) \neq 0$, $\Im(I(\gamma)) = 0$ and the lattice points corresponding to the subsequence $\{n\} \subseteq S$ converge to

$$\mu_n \Rightarrow \frac{1}{4}\delta_{\pi/4} \star (\delta_{\pm 1} + \delta_{\pm i})$$

the tilted Cilleruelo measure, i.e. Cilleruelo measure rotated by $\frac{\pi}{4}$.

In order to analyse either of the scenarios (1) and (2) one needs to understand which of the terms

$$\frac{n}{\mathcal{N}_n} \cdot (4B_{\mathcal{C}} - L^2)$$

(leading term of the 2nd chaotic projection, see (4.6)) or $\frac{n}{\mathcal{N}_n^2}$ (the order of magnitude of the next term) is dominant by order of magnitude, knowing that in either case (1) or (2), leading coefficient $4B_{\mathcal{C}} - L^2$ vanishes asymptotically, at least under the δ -separatedness assumption. Equivalently, whether $4B_{\mathcal{C}} - L^2$ vanishes more rapidly than $\frac{1}{\mathcal{N}_n}$; this would most certainly involve the rate of convergence of μ_n , and it is plausible that one can construct sequences $\{n\}$ observing both kinds of behaviour.

Towards the end of the introduction we would like to discuss the possibility of extending the presented results to higher dimensional tori (that is, the distribution of the nodal intersections number of higher dimensional arithmetic random waves against a smooth curve). In this case the associated lattice points problems are significantly more delicate, and we are not aware of precise asymptotic results concerning the variance, let alone the more subtle question of the limit law, with only a few partial results published [RWY16, Ma17], for non-vanishing curvature curves and straight segments respectively. Finally we use this opportunity to point out the recent work [CNNV17] on the universality of various statistics related to the nodal intersections number (such as its mean and higher moments) w.r.t. the law of random coefficients a_λ in (1.1).

²We wish to thank Michael Benedicks for pointing out [OY05] to us.

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2. OUTLINE OF THE PAPER

2.1. On the proof of the main results. The proofs of theorems 1.1 and 1.3 are based on the *chaotic expansion* for the nodal intersections number (see §2.2 and Appendix A). We first consider a unit speed parametrization of the curve $\gamma : [0, L] \rightarrow \mathcal{C}$, and set

$$(2.1) \quad f_n : [0, L] \rightarrow \mathbb{R}; \quad t \mapsto T_n(\gamma(t)).$$

The map f_n in (2.1) defines a (non-stationary) centered Gaussian process on $[0, L]$ with covariance function

$$(2.2) \quad r_n(t_1, t_2) := \text{Cov}(f_n(t_1), f_n(t_2)) = \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \cos(2\pi \langle \lambda, \gamma(t_1) - \gamma(t_2) \rangle), \quad t_1, t_2 \in [0, L],$$

see (1.4). The number \mathcal{Z}_n of nodal intersections of T_n against \mathcal{C} equals to the number of zero crossings of f_n in the interval $[0, L]$, and we can formally write

$$(2.3) \quad \mathcal{Z}_n = \int_0^L \delta_0(f_n(t)) |f'_n(t)| dt,$$

where δ_0 is the Dirac delta function.

The random variable \mathcal{Z}_n in (2.3) admits the Wiener-Îto chaotic expansion (see §2.2 and Appendix A) of the form

$$(2.4) \quad \mathcal{Z}_n = \sum_{q=0} \mathcal{Z}_n[2q],$$

where the above series converges in the space $L^2(\mathbb{P})$ of random variables with finite variance. In particular, the random variables $\mathcal{Z}_n[2q]$, $\mathcal{Z}_n[2q']$ are orthogonal (uncorrelated) for $q \neq q'$, and $\mathcal{Z}_n[0] = \mathbb{E}[\mathcal{Z}_n]$.

Evaluating the second chaotic projection $\mathcal{Z}_n[2]$ yields that, under the assumptions in Theorem 1.1, the variance of the total number \mathcal{Z}_n of nodal intersections is asymptotic to the variance of $\mathcal{Z}_n[2]$, both being asymptotic to (1.10). The latter and the orthogonality of the Wiener chaoses imply that the distribution of $\mathcal{Z}_n[2]$ dominates the series on the

r.h.s. of (2.4), and a Central Limit Theorem result for $\mathcal{Z}_n[2]$ allows to infer the statement of Theorem 1.1.

Assume now that the curve \mathcal{C} is static (Definition 1). The leading term in (1.10) vanishes, and we are left only with an upper bound for the variance of \mathcal{Z}_n . To obtain its precise asymptotics we need to inspect the proof of the *approximate* Kac-Rice formula [RW16, Proposition 1.3], and obtain one extra term in the Taylor expansion of the two-point correlation function (see (5.1) in §5). The main difficulty is how to control “off-diagonal” terms coming from the fourth moment of the covariance function r_n in (1.4), and specifically the quantity

$$(2.5) \quad \frac{1}{\mathcal{N}_n^2} \sum_{\substack{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Lambda_n \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \neq 0}} \frac{1}{\|\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\|},$$

required to be evaluated in order to control the fourth moment of r_n . We will use some properties of the δ -separated sequences of energy levels (Definition 2) to show that the expression in (2.5) is $o(1)$ (see Lemma 5.2), and then prove (1.23).

It turns out that the leading term in the chaotic expansion (2.4) is no longer the projection onto the second chaos, but the projection $\mathcal{Z}_n[4]$ onto the fourth chaos. A precise analysis of the latter allows to get its asymptotic (non-Gaussian) distribution in (1.20), thus concluding the proof of Theorem 1.3 by a standard application of [Dud02, Theorem 11.7.1].

2.2. Chaos expansion. In this section we compute the chaotic expansion (2.4) for the nodal intersections number \mathcal{Z}_n . The reader can refer to Appendix A, where the discussion is focussed on our particular situation, or [NP12], for a complete discussion on Wiener-Îto chaos expansions. Recall the definition (2.1) of the random process f_n and the formal expression (2.3). Note that for every $t \in [0, L]$

$$f'_n(t) = \langle \nabla T_n(\gamma(t)), \dot{\gamma}(t) \rangle,$$

where ∇T_n denotes the gradient of T_n and $\dot{\gamma}$ the first derivative of γ . We have [RW16, Lemma 1.1] that

$$(2.6) \quad \text{Var}(f'_n(t)) = 2\pi^2 n =: \alpha.$$

We can then rewrite (2.3) as

$$(2.7) \quad \mathcal{Z}_n = \sqrt{2\pi^2 n} \int_0^L \delta_0(f_n(t)) |\tilde{f}'_n(t)| dt,$$

where

$$\tilde{f}'_n(t) := \frac{f'_n(t)}{\sqrt{2\pi^2 n}}, \quad t \in [0, L].$$

As f_n is a unit variance process, differentiating the equality $r(t, t) \equiv 1$ yields that for every $t \in [0, L]$, $f_n(t)$ and $f'_n(t)$ are independent (see e.g. [RW16, Lemma 2.2]), and so are $f_n(t)$ and $\tilde{f}'_n(t)$. Heuristically, in order to find the chaotic expansion for \mathcal{Z}_n in (2.7) we first compute the chaotic decompositions³ for $\delta_0(Z)$ and $|W|$, where (Z, W) is a standard Gaussian random vector. Since both the Dirac mass at 0 and the absolute value are even

³Note that the chaotic decomposition of $\delta_0(Z)$ is just a *formal* expansion

functions, in these expansions only terms corresponding to even order Wiener chaoses appear:

$$(2.8) \quad \delta_0(Z) = \sum_{q=0}^{+\infty} b_{2q} H_{2q}(Z), \quad |W| = \sum_{\ell=0}^{+\infty} a_{2\ell} H_{2\ell}(W),$$

where $\{H_k, k = 0, 1, \dots\}$ denotes the Hermite polynomials (see [Sze75, §5.5] and Appendix A) and the coefficients $b_{2k}, a_{2k}, k = 0, 1, 2, \dots$ are given by (see e.g. [KL97]):

$$(2.9) \quad b_{2q} := \frac{1}{(2q)! \sqrt{2\pi}} H_{2q}(0), \quad q \geq 0,$$

and

$$(2.10) \quad a_{2\ell} := \sqrt{\frac{2}{\pi}} \frac{(-1)^{\ell+1}}{2^\ell \ell! (2\ell - 1)}, \quad \ell \geq 0.$$

In order to obtain the (formal) chaotic expansion of $\delta_0(Z)|W|$, the random variables Z and W being independent, it suffices to “multiply” the two series in (2.8): the $2q$ -th chaotic component is given by the sum of all terms of the form $b_{2m} a_{2n} H_{2m}(Z) H_{2n}(W)$ for $m + n = q$ (see the so-called product formula [NP12, Theorem 2.7.10]). To obtain the chaotic expansion for \mathcal{Z}_n in (2.7), on heuristic level, we replace (Z, W) by $(f_n(t), \tilde{f}'_n(t))$ and then integrate over $t \in [0, L]$ the (formal) series expansion for $\delta_0(f_n(t))|\tilde{f}'_n(t)|$.

To be more precise, from [KL97, Lemma 2] we have the chaotic expansion (2.4) for the nodal intersection number (2.7)

$$(2.11) \quad \mathcal{Z}_n = \sum_{q=0}^{+\infty} \mathcal{Z}_n[2q] = \sqrt{2\pi^2 n} \sum_{q=0}^{+\infty} \sum_{\ell=0}^q b_{2q-2\ell} a_{2\ell} \int_0^L H_{2q-2\ell}(f_n(t)) H_{2\ell}(\tilde{f}'_n(t)) dt;$$

in particular from (2.11) we have for $q \geq 0$

$$(2.12) \quad \mathcal{Z}_n[2q] = \sqrt{2\pi^2 n} \sum_{\ell=0}^q b_{2q-2\ell} a_{2\ell} \int_0^L H_{2q-2\ell}(f_n(t)) H_{2\ell}(\tilde{f}'_n(t)) dt.$$

2.3. Plan of the paper. In §3 we will state a few key propositions instrumental in proving theorems 1.1 and 1.3, in particular, concerning $\mathcal{Z}_n[2]$ in (2.12) for “generic” curves (for Theorem 1.1), and $\mathcal{Z}_n[4]$, and also the “approximate Kac-Rice formula” (5.1) for static curves (for Theorem 1.3). In §4 we will then investigate the second chaotic component for “generic” curves, whereas in §6 the fourth one in the case of static curves. The proof of the approximate Kac-Rice formula for the variance of the number of nodal intersections against static curves will be given in §5.

In the Appendix we will collect some background and several technical results (concerning chaotic components, approximate Kac-Rice formula and bounds for certain summations over lattice points such as (2.5)) and, in particular, in §G a family of static curves will be constructed.

3. PROOFS OF THE MAIN RESULTS

The zeroth term in the chaos expansion (2.11) of \mathcal{Z}_n is the orthogonal projection of \mathcal{Z}_n onto the zeroth Wiener chaos (i.e., onto \mathbb{R}), hence it coincides with the mean of the random variable itself, consistently with the following lemma (cf. (1.5)).

Lemma 3.1. *For every $n \in S$,*

$$(3.1) \quad \mathcal{Z}_n[0] = \frac{\sqrt{E_n}}{\pi\sqrt{2}} L.$$

Proof. From (2.9), (2.10) and (2.12) we have, for $q = 0$,

$$\mathcal{Z}_n[0] = \sqrt{2\pi^2 n} b_0 a_0 L = \sqrt{2\pi^2 n} \frac{1}{\sqrt{2\pi}} \sqrt{\frac{2}{\pi}} L = \sqrt{2n} L = \frac{\sqrt{E_n}}{\pi\sqrt{2}} L.$$

□

3.1. Proof of Theorem 1.1. To prove Theorem 1.1, we first need to study the asymptotic variance of the second chaotic component $\mathcal{Z}_n[2]$. It turns out that, under the scenario covered by Theorem 1.1, the second chaotic component $\mathcal{Z}_n[2]$ dominates the distribution of \mathcal{Z}_n (a by-product of Proposition 3.2 below and the orthogonality of the Wiener chaos spaces), and the convergence in distribution of $\mathcal{Z}_n[2]$ to Gaussian (Proposition 3.3) will allow us to deduce the same for \mathcal{Z}_n .

Proposition 3.2. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth curve on the torus with nowhere zero curvature, of total length L , and $\{n\} \subset S$ such that $\mathcal{N}_n \rightarrow +\infty$, and $\{4B_{\mathcal{C}}(\mu_n) - L^2\}$ is bounded away from zero. Then*

$$\text{Var}(\mathcal{Z}_n) \sim \text{Var}(\mathcal{Z}_n[2]),$$

i.e. the variance of $\mathcal{Z}_n[2]$ is asymptotic to the variance (1.10) of the nodal intersections \mathcal{Z}_n .

In light of Proposition 3.2 to be proven in §4, we are to study the asymptotic distribution of the second chaotic component.

Proposition 3.3. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth curve on the torus with nowhere zero curvature, of total length L , and $\{n\} \subset S$ such that $\mathcal{N}_n \rightarrow +\infty$, and $\{4B_{\mathcal{C}}(\mu_n) - L^2\}$ is bounded away from zero. Then*

$$\frac{\mathcal{Z}_n[2]}{\sqrt{\text{Var}(\mathcal{Z}_n[2])}} \xrightarrow{d} Z,$$

where $Z \sim \mathcal{N}(0, 1)$.

Proposition 3.3 will be proven in §4. We are in a position to prove our first main result.

Proof of Theorem 1.1 assuming propositions 3.2 and 3.3. By (2.11) we write

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} = \sum_{q=1}^{+\infty} \frac{\mathcal{Z}_n[2q]}{\sqrt{\text{Var}(\mathcal{Z}_n)}}.$$

Thanks to Proposition 3.2 and the orthogonality of Wiener chaoses we have, as $\mathcal{N}_n \rightarrow \infty$,

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} = \frac{\mathcal{Z}_n[2]}{\sqrt{\text{Var}(\mathcal{Z}_n[2])}} + o_{\mathbb{P}}(1),$$

where $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to zero in probability. In particular, the distribution of the normalized total number of nodal intersections is asymptotic to the distribution of $\frac{\mathcal{Z}_n[2]}{\sqrt{\text{Var}(\mathcal{Z}_n[2])}}$. The latter and Proposition 3.3 imply the statement of Theorem 1.1.

□

3.2. Proof of Theorem 1.3. It turns out that, under the scenario covered by Theorem 1.3, the 4th chaotic component $\mathcal{Z}_n[4]$ dominates the distribution of \mathcal{Z}_n (a by-product of propositions 3.4 and 3.5 below, and the orthogonality of the Wiener chaos spaces). The distribution law of \mathcal{Z}_n is then asymptotic to the one of $\mathcal{Z}_n[4]$ in Proposition 3.6. Though in this scenario, the 2nd chaotic component $\mathcal{Z}_n[2]$ does not vanish precisely, it does not contribute to the asymptotic law of \mathcal{Z}_n (also follows from Proposition 3.4). Unlike propositions 3.2-3.3, to prove the more delicate propositions 3.4-3.5 we will invoke both the assumptions on \mathcal{C} and $\{n\}$, i.e. \mathcal{C} is static and $\{n\}$ is δ -separated. Our first proposition asserts the variance part (1.23) of Theorem 1.3, to be proven in §5.

Proposition 3.4. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L , and $\{n\} \subset S$ be a δ -separated sequence such that $\mathcal{N}_n \rightarrow +\infty$. Then*

$$(3.2) \quad \text{Var}(\mathcal{Z}_n) = \frac{n}{4\mathcal{N}_n^2} (16A_{\mathcal{C}}(\mu_n) - L^2) (1 + o(1)),$$

where $A_{\mathcal{C}}(\mu_n)$ is given in (1.19) with $\mu = \mu_n$. Moreover, the leading term $16A_{\mathcal{C}}(\mu_n) - L^2$ in (3.2) is bounded away from zero.

Next we assert that, under the assumptions of Theorem 1.3, the fourth term in the chaotic series (2.11) dominates.

Proposition 3.5. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L , and $\{n\} \subset S$ a δ -separated sequence such that $\mathcal{N}_n \rightarrow +\infty$. Then*

$$(3.3) \quad \text{Var}(\mathcal{Z}_n[2]) = o\left(\frac{n}{\mathcal{N}_n^2}\right),$$

and

$$(3.4) \quad \text{Var}(\mathcal{Z}_n[4]) \sim \frac{n}{4\mathcal{N}_n^2} (16A_{\mathcal{C}}(\mu_n) - L^2).$$

The above implies that it suffices to study the asymptotic distribution of the fourth chaotic projection.

Proposition 3.6. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L , and $\{n\} \subset S$ a δ -separated sequence such that $\mathcal{N}_n \rightarrow +\infty$, and $\mu_n \Rightarrow \mu$. Then*

$$(3.5) \quad \frac{\mathcal{Z}_n[4]}{\sqrt{\text{Var}(\mathcal{Z}_n[4])}} \xrightarrow{d} \mathcal{M}(\mu),$$

where $\mathcal{M}(\mu)$ is given by (1.20).

Proof of Theorem 1.3 assuming propositions 3.4-3.6. By (2.11) we write

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} = \sum_{q=1}^{+\infty} \frac{\mathcal{Z}_n[2q]}{\sqrt{\text{Var}(\mathcal{Z}_n)}}.$$

Thanks to Proposition 3.4, Proposition 3.5 and the orthogonality of Wiener chaoses we have, as $\mathcal{N}_n \rightarrow \infty$,

$$\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}} = \frac{\mathcal{Z}_n[4]}{\sqrt{\text{Var}(\mathcal{Z}_n[4])}} + o_{\mathbb{P}}(1),$$

where $o_{\mathbb{P}}(1)$ denotes convergence to zero in probability. In particular, the distribution of the normalized total number of nodal intersections is asymptotic to the one of

$$\frac{\mathcal{Z}_n[4]}{\sqrt{\text{Var}(\mathcal{Z}_n[4])}}.$$

Therefore (3.5) also holds with \mathcal{Z}_n in place of $\mathcal{Z}_n[4]$. In particular, this implies

$$d\left(\frac{\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]}{\sqrt{\text{Var}(\mathcal{Z}_n)}}, \mathcal{M}(\mu_n)\right) \rightarrow 0,$$

where d is any metric which metrizes convergence in distribution of random variables, or the Kolmogorov distance (see e.g. [NP12, §C]), since $\mathcal{M}(\mu)$ in (1.20) has absolutely continuous distribution for arbitrary probability measure μ . Theorem 1.3 is then a direct consequence of [Dud02, Theorem 11.7.1] and of the fact that the sequence $\left\{(\mathcal{Z}_n - \mathbb{E}[\mathcal{Z}_n]) / \sqrt{\text{Var}(\mathcal{Z}_n)}\right\}$ is bounded in $L^2(\mathbb{P})$. □

4. PROOFS OF PROPOSITION 3.2 AND PROPOSITION 3.3

In this section we investigate the asymptotic behaviour of the second chaotic component $\mathcal{Z}_n[2]$ of \mathcal{Z}_n . Our starting point is the decomposition of $\mathcal{Z}_n[2]$ into “diagonal” and “off-diagonal” terms: we define the diagonal term

$$(4.1) \quad \mathcal{Z}_n^a[2] := \frac{\sqrt{2\pi^2 n}}{2\pi} \frac{1}{\mathcal{N}_n} 2 \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1) \left(2 \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt - L \right),$$

where if \sqrt{n} is not an integer $\Lambda_n^+ := \{\lambda \in \Lambda_n : \lambda_2 > 0\}$, otherwise

$$\Lambda_n^+ := \{\lambda \in \Lambda_n : \lambda_2 > 0\} \cup \{(\sqrt{n}, 0)\}.$$

The off-diagonal term is

$$(4.2) \quad \mathcal{Z}_n^b[2] := \frac{\sqrt{2\pi^2 n}}{2\pi} \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L \left(2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle - 1 \right) e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt.$$

Recall that, by construction, $\mathcal{Z}_n[2]$, $\mathcal{Z}_n^a[2]$ and $\mathcal{Z}_n^b[2]$ are all mean zero, so their variances are equal to their respective second moments.

Lemma 4.1. *For every $n \in S$ we have*

$$(4.3) \quad \mathcal{Z}_n[2] = \mathcal{Z}_n^a[2] + \mathcal{Z}_n^b[2].$$

The proof of Lemma 4.1 will be given in Appendix B. We are now in a position to give a proof for Proposition 3.2; first it is useful to recall the following identity:

Lemma 4.2 ([RW08, Lemma 2.3]). *For every $z \in \mathbb{R}^2$ we have the following equality:*

$$(4.4) \quad \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \left\langle z, \frac{\lambda}{|\lambda|} \right\rangle = \frac{1}{2} |z|^2.$$

Proof of Proposition 3.2 assuming Lemma 4.1. Lemma 4.1 yields

$$(4.5) \quad \text{Var}(\mathcal{Z}_n[2]) = \text{Var}(\mathcal{Z}_n^a[2]) + \text{Var}(\mathcal{Z}_n^b[2]) + 2\text{Cov}(\mathcal{Z}_n^a[2], \mathcal{Z}_n^b[2]).$$

Let us first study $\text{Var}(\mathcal{Z}_n^a[2])$. By the definition (4.1) of $\mathcal{Z}_n^a[2]$ we may compute its variance to be

$$(4.6) \quad \begin{aligned} \text{Var}(\mathcal{Z}_n^a[2]) &= 2 \frac{n}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n^+} \left(2 \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt - L \right)^2 \\ &= \frac{n}{\mathcal{N}_n^2} \sum_{\lambda \in \Lambda_n} \left(2 \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt - L \right)^2 \\ &= \frac{n}{\mathcal{N}_n} \left(4 \int_0^L \int_0^L \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_1) \right\rangle^2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_2) \right\rangle^2 dt_1 dt_2 \right. \\ &\quad \left. - 4L \int_0^L \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt + L^2 \right) \\ &= \frac{n}{\mathcal{N}_n} (4B_C(\mu_n) - 2L^2 + L^2) = \frac{n}{\mathcal{N}_n} (4B_C(\mu_n) - L^2), \end{aligned}$$

where we used (4.4) with $z = \dot{\gamma}(t)$ to get from the 4th to the 5th line of (4.6). Next we evaluate the variance of $\mathcal{Z}_n^b[2]$: by (4.2) it is given by

$$(4.7) \quad \begin{aligned} \text{Var}(\mathcal{Z}_n^b[2]) &= \frac{n}{2\mathcal{N}_n^2} \sum_{\lambda \neq \lambda', \lambda'' \neq \lambda'''} \mathbb{E}[a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}}] \times \\ &\quad \times \int_0^L \left(2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle - 1 \right) e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \times \\ &\quad \times \int_0^L \left(2 \left\langle \frac{\lambda''}{|\lambda''|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'''}{|\lambda'''|}, \dot{\gamma}(t) \right\rangle - 1 \right) e^{i2\pi \langle \lambda'' - \lambda''', \gamma(t) \rangle} dt \\ &\leq C \frac{n}{2\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \left| \int_0^L \left(2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle - 1 \right) e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \right|^2, \end{aligned}$$

for some $C > 0$.

Using the non-vanishing curvature assumption on \mathcal{C} , Van der Corput's Lemma (see e.g. [RW16, Lemma 5.2]) implies that for $\lambda \neq \lambda'$ the inner oscillatory integral on the r.h.s. of (4.7) may be bounded as

$$\left| \int_0^L \left(2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle - 1 \right) e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \right| \ll \frac{1}{|\lambda - \lambda'|^{1/2}};$$

this together with (4.7) yield

$$(4.8) \quad \text{Var}(\mathcal{Z}_n^b[2]) \ll \frac{n}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \frac{1}{|\lambda - \lambda'|}.$$

Using the bound

$$\sum_{\lambda \neq \lambda'} \frac{1}{|\lambda - \lambda'|} \ll_\epsilon \mathcal{N}_n^\epsilon$$

for every $\epsilon > 0$ from [RW16, Proposition 5.3] to bound the r.h.s. of (4.8), and comparing the result with (4.6) we have

$$(4.9) \quad \text{Var}(\mathcal{Z}_n^b[2]) = o(\text{Var}(\mathcal{Z}_n^a[2])).$$

Now substituting (4.6), (4.9) into (4.5), and using Cauchy-Schwarz to bound the covariance term in (4.5), finally yield

$$\text{Var}(\mathcal{Z}_n[2]) \sim \text{Var}(\mathcal{Z}_n[2]^a) = \frac{n}{\mathcal{N}_n}(4B_C(\mu_n) - L^2).$$

This taking into account (1.10), concludes the proof of Proposition 3.2. \square

Proof of Proposition 3.3 assuming Lemma 4.1. Thanks to (4.9), it suffices to investigate the asymptotic distribution of $\mathcal{Z}_n^a[2]$ as in (4.1). We have

$$(4.10) \quad \frac{\mathcal{Z}_n^a[2]}{\sqrt{\text{Var}(\mathcal{Z}_n^a[2])}} = \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1) \frac{2 \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt - L}{\sqrt{4B_C(\mu_n) - L^2}}.$$

Now we can apply Lindeberg's criterion (see e.g. [NP12, Remark 11.1.2]): since, as $\mathcal{N}_n \rightarrow +\infty$, we have

$$\max_{\lambda \in \Lambda_n^+} \frac{1}{\sqrt{\mathcal{N}_n/2}} \left| \frac{2 \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt - L}{\sqrt{4B_C(\mu_n) - L^2}} \right| \rightarrow 0,$$

then

$$\frac{\mathcal{Z}_n^a[2]}{\sqrt{\text{Var}(\mathcal{Z}_n^a[2])}} \xrightarrow{d} Z,$$

where $Z \sim \mathcal{N}(0, 1)$. \square

5. PROOF OF PROPOSITION 3.4

The proof of Proposition 3.4 is inspired by the proof of Theorem 1.2 in [RW16], we refer the reader to [RW16, §1.3] and [RWY16, §1.5] for a complete discussion. We will need to inspect the proof of the *approximate* Kac-Rice formula [RW16, Proposition 1.3], which gives the asymptotic variance of the nodal intersections number in terms of an explicit integral that involves the covariance function $r = r_n$ in (2.2) and a couple of its derivatives

$$r_1 := \frac{\partial}{\partial t_1} r, \quad r_2 := \frac{\partial}{\partial t_2} r, \quad r_{12} := \frac{\partial^2}{\partial t_1 \partial t_2} r,$$

and then study higher order terms.

5.1. Auxiliary lemmas. The following lemma, proved in §5.3 below, referred to as “Approximate Kac-Rice formula” (due to the existence of the error term on the r.h.s. of 5.1), and valid for static curves only, is a refined version of [RW16, Proposition 1.3], with more terms in the main expansion, and a smaller error term. Recall that α is given by (2.6).

Lemma 5.1 (“Approximate Kac-Rice”). *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of total length L , and $\{n\} \subset S$ a δ -separated sequence such that $\mathcal{N}_n \rightarrow +\infty$. Then the intersection number variance is asymptotic to*

$$\begin{aligned}
 \text{Var}(\mathcal{Z}_n) = n \int_0^L \int_0^L & \left(\frac{3}{4}r^4 + \frac{1}{12}(r_{12}/\alpha)^4 - \frac{(r_2/\sqrt{\alpha})^4}{4} - \frac{(r_1/\sqrt{\alpha})^4}{4} \right. \\
 & + 2(r_{12}/\alpha)r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) + \frac{(r_1/\sqrt{\alpha})^2(r_2/\sqrt{\alpha})^2}{2} - \frac{3}{2}r^2(r_2/\sqrt{\alpha})^2 \\
 & - \frac{3}{2}r^2(r_1/\sqrt{\alpha})^2 + \frac{1}{2}(r_{12}/\alpha)^2r^2 + \frac{1}{2}(r_2/\sqrt{\alpha})^2(r_{12}/\alpha)^2 \\
 & \left. + \frac{1}{2}(r_1/\sqrt{\alpha})^2(r_{12}/\alpha)^2 \right) dt_1 dt_2 + o\left(\frac{n}{\mathcal{N}_n^2}\right).
 \end{aligned}
 \tag{5.1}$$

To find the asymptotics of the moments of r and its derivatives appearing in the r.h.s. of (5.1), in particular to bound the contribution of “off-diagonal” terms, we will need the following lemma whose proof is given in Appendix E.

Lemma 5.2. *Assume that $\{n\} \subset S$ is a δ -separated sequence, then*

$$\min_{\substack{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Lambda_n \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \neq 0}} \|\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\| \gg n^{2\delta},$$

where the constant involved in the “ \gg ” notation is absolute. In particular,

$$\frac{1}{\mathcal{N}_n^2} \sum_{\substack{\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in \Lambda_n \\ \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 \neq 0}} \frac{1}{\|\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4\|} = o(1).
 \tag{5.2}$$

Let us also introduce some more notation:

$$F_{\mathcal{C}}(\mu_n) := \int_0^L \int_0^L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda'} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_2) \right\rangle dt_1 dt_2.
 \tag{5.3}$$

We can now state the following lemma, whose proof will be given in Appendix D below.

Lemma 5.3. *If $\mathcal{C} \subset \mathbb{T}$ is a smooth curve with nowhere vanishing curvature, then for δ -separated sequences $\{n\}$ such that $\mathcal{N}_n \rightarrow +\infty$, we have*

$$\begin{aligned}
(5.4) \quad & 1) \int_{\mathcal{C}} \int_{\mathcal{C}} r(t_1, t_2)^4 dt_1 dt_2 = 3L^2 \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}), \\
& 2) \int_{\mathcal{C}} \int_{\mathcal{C}} \left(\frac{1}{\sqrt{\alpha}} r_1(t_1, t_2) \right)^4 dt_1 dt_2 = 3L^2 \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}), \\
& 3) \int_{\mathcal{C}} \int_{\mathcal{C}} \left(\frac{1}{\alpha} r_{12}(t_1, t_2) \right)^4 dt_1 dt_2 = 2^4 \cdot 3A_{\mathcal{C}}(\mu_n) \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}), \\
& 4) \int_{\mathcal{C}} \int_{\mathcal{C}} (r_{12}/\alpha) r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) dt_1 dt_2 = -4F_{\mathcal{C}}(\mu_n) \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}), \\
& 5) \int_{\mathcal{C}} \int_{\mathcal{C}} (r_1/\sqrt{\alpha})^2 (r_2/\sqrt{\alpha})^2 dt_1 dt_2 = L^2 \frac{1}{\mathcal{N}_n^2} + 4 \cdot 2 \frac{1}{\mathcal{N}_n^2} F_{\mathcal{C}}(\mu_n) + o(\mathcal{N}_n^{-2}), \\
& 6) \int_{\mathcal{C}} \int_{\mathcal{C}} r^2 (r_2/\sqrt{\alpha})^2 dt_1 dt_2 = \frac{1}{\mathcal{N}_n^2} L^2 + o(\mathcal{N}_n^{-2}), \\
& 7) \int_{\mathcal{C}} \int_{\mathcal{C}} r^2 (r_{12}/\alpha)^2 dt_1 dt_2 = 4 \frac{1}{\mathcal{N}_n^2} B_{\mathcal{C}}(\mu_n) + 8 \frac{1}{\mathcal{N}_n^2} F_{\mathcal{C}}(\mu_n) + o(\mathcal{N}_n^{-2}), \\
& 8) \int_{\mathcal{C}} \int_{\mathcal{C}} (r_1/\sqrt{\alpha})^2 (r_{12}/\alpha)^2 dt_1 dt_2 = 4 \frac{1}{\mathcal{N}_n^2} B_{\mathcal{C}}(\mu_n) + o(\mathcal{N}_n^{-2}),
\end{aligned}$$

where $B_{\mathcal{C}}(\mu_n)$, $A_{\mathcal{C}}(\mu_n)$ are given in (1.11) and (1.19) with $\mu = \mu_n$, respectively, and $F_{\mathcal{C}}(\mu_n)$ in (5.3).

5.2. Proof of Proposition 3.4.

Proof. Upon substituting (5.4) into (5.1), we obtain

$$(5.5) \quad \text{Var}(\mathcal{Z}_n) = \frac{n}{4\mathcal{N}_n^2} (16A_{\mathcal{C}}(\mu_n) + 24B_{\mathcal{C}}(\mu_n) - 7L^2) + o\left(\frac{n}{\mathcal{N}_n^2}\right)$$

after some straightforward manipulations. Since \mathcal{C} is static, $4B_{\mathcal{C}}(\mu_n) = L^2$ so that (5.5) is

$$\text{Var}(\mathcal{Z}_n) = \frac{n}{4\mathcal{N}_n^2} (16A_{\mathcal{C}}(\mu_n) - L^2) + o\left(\frac{n}{\mathcal{N}_n^2}\right),$$

which is (3.2).

Now we prove that the leading term $16A_{\mathcal{C}}(\mu_n) - L^2$ is bounded away from zero, assuming that \mathcal{C} is static. In what follows, we show that if \mathcal{C} is static, then $16A_{\mathcal{C}}(\mu) - L^2 > 0$ is strictly positive for every probability measure on \mathcal{S}^1 invariant w.r.t. rotation by $\frac{\pi}{2}$; by the compactness of $\frac{\pi}{2}$ -invariant probability measures this is sufficient for our purposes.

By denoting the inner integral in (1.19)

$$B(\theta_1, \theta_2) := \int_0^L \langle \theta_1, \dot{\gamma}(t) \rangle^2 \langle \theta_2, \dot{\gamma}(t) \rangle^2 dt,$$

we may rewrite the definition (1.19) of $A_{\mathcal{C}}(\mu)$ as

$$A_{\mathcal{C}}(\mu) = \int_{\mathcal{S}^1} \int_{\mathcal{S}^1} B(\theta_1, \theta_2)^2 d\mu(\theta_1) d\mu(\theta_2),$$

For $\theta \in \mathcal{S}^1$ we denote θ^\perp to be its rotation by $\frac{\pi}{2}$. Then, for every $\theta_1, \theta_2 \in \mathcal{S}^1$, we have

$$(5.6) \quad B(\theta_1, \theta_2) + B(\theta_1^\perp, \theta_2) + B(\theta_1, \theta_2^\perp) + B(\theta_1^\perp, \theta_2^\perp) = L.$$

Upon maximizing and minimizing the function $B(\theta_1, \theta_2)^2 + B(\theta_1^\perp, \theta_2)^2 + B(\theta_1, \theta_2^\perp)^2 + B(\theta_1^\perp, \theta_2^\perp)^2$ under the constraint (5.6), we get that

$$(5.7) \quad \frac{L^2}{4} \leq B(\theta_1, \theta_2)^2 + B(\theta_1^\perp, \theta_2)^2 + B(\theta_1, \theta_2^\perp)^2 + B(\theta_1^\perp, \theta_2^\perp)^2 \leq L^2, \quad \forall \theta_1, \theta_2 \in \mathcal{S}^1.$$

We can exploit the invariance property of the probability measure μ and of $B(\cdot, \cdot)$ w.r.t. sign changes, to rewrite (5.2) as

$$A_C(\mu) = \int_{\mathcal{S}^1/i} \int_{\mathcal{S}^1/i} 4 (B(\theta_1, \theta_2)^2 + B(\theta_1^\perp, \theta_2)^2 + B(\theta_1, \theta_2^\perp)^2 + B(\theta_1^\perp, \theta_2^\perp)^2) d\mu(\theta_1) d\mu(\theta_2),$$

where \mathcal{S}^1/i is a quarter of the circle of measure $\mu(\mathcal{S}^1/i) = 1/4$. By (5.7) we have

$$\frac{L^2}{16} \leq A_C(\mu) \leq \frac{L^2}{4},$$

which implies

$$0 \leq 16A_C(\mu) - L^2 \leq 3L^2,$$

and moreover, that

$$A_C(\mu) = \frac{L^2}{16},$$

if and only if for every $\theta_1, \theta_2 \in \text{supp } \mu$ in the support of μ , we have

$$B(\theta_1, \theta_2)^2 + B(\theta_1^\perp, \theta_2)^2 + B(\theta_1, \theta_2^\perp)^2 + B(\theta_1^\perp, \theta_2^\perp)^2 = \frac{L^2}{4}.$$

Further, given $\theta_1, \theta_2 \in \mathcal{S}^1$, the minimum in (5.7) is attained, i.e.

$$B(\theta_1, \theta_2)^2 + B(\theta_1^\perp, \theta_2)^2 + B(\theta_1, \theta_2^\perp)^2 + B(\theta_1^\perp, \theta_2^\perp)^2 = \frac{L^2}{4},$$

if and only if

$$B(\theta_1, \theta_2) = B(\theta_1^\perp, \theta_2) = B(\theta_1, \theta_2^\perp) = B(\theta_1^\perp, \theta_2^\perp) = \frac{L}{4}.$$

Now assume that for some μ invariant w.r.t. $\frac{\pi}{2}$ -rotation, we have

$$A_C(\mu) = \frac{L^2}{16},$$

so that, by the above, we have $B(\theta_0, \theta_0) = \frac{L}{4}$ for some $\theta_0 \in \mathcal{S}^1$ (any $\theta_0 \in \text{supp } \mu$ would do). For a unit speed parametrization $\gamma : [0, L] \rightarrow \mathcal{C}$ we have $\dot{\gamma}(t) = e^{i\phi(t)}$, where $\phi(t)$ is the angle of the derivative $\dot{\gamma}(t)$ w.r.t. the coordinate axis. Recalling standard trigonometric identities, we may write

$$(5.8) \quad \begin{aligned} \frac{L}{4} &= B(\theta_0, \theta_0) = \int_0^L \cos^4(\phi(t) - \theta_0) dt \\ &= \frac{3}{8}L + \frac{1}{2} \int_0^L \cos(2(\phi(t) - \theta_0)) dt + \frac{1}{8} \int_0^L \cos(4(\phi(t) - \theta_0)) dt. \end{aligned}$$

From [RW16, Corollary 7.2] for static curves \mathcal{C} we have

$$(5.9) \quad \int_0^L \cos(2(\phi(t) - \theta_0)) dt = 0,$$

and substituting (5.9) into (5.8), we obtain

$$(5.10) \quad \int_0^L \cos(4(\phi(t) - \theta_0)) dt = -L.$$

The equality (5.10) holds true if and only if for every $t \in [0, L]$

$$\cos(4(\phi(t) - \theta_0)) = -1,$$

which certainly implies that \mathcal{C} is a straight line segment, which, by our conventions (Definition 1), is a contradiction to our assumption that \mathcal{C} is static. \square

The rest of this section is dedicated to proving Lemma 5.1.

5.3. Proof of Lemma 5.1. Our proof follows along the same path of thought as [RW16, Proposition 1.3], except that we add one more term in the expansion of the 2-point correlation function (5.16), and need to control the error terms using the more difficult to evaluate higher moments. Thereupon we are going to bring the essence of the proof, highlighting the differences, sometimes omitting some details identical to both cases.

5.3.1. Preliminaries. The main idea is to divide the square $[0, L]^2$ into small sub-squares and apply the usual Kac-Rice formula to “most” of them, bounding the contribution of the remaining terms (for an extensive discussion see [RW16, §1.3] and [RWY16, §1.5]).

Let us divide the interval $[0, L]$ into small sub-intervals of length roughly $1/\sqrt{E_n}$. To be more precise, let $c_0 > 0$ (chosen as in Lemma C.1) and set $k := \lfloor L\sqrt{E_n}/c_0 \rfloor + 1$ and $\delta_0 := L/k$; consider the sub-intervals, for $i = 1, \dots, k$, defined as

$$I_i := [(i-1)\delta_0, i\delta_0].$$

Let us now denote by \mathcal{Z}_i the number of zeros of f_n in I_i , so that

$$\mathcal{Z}_n = \sum_i \mathcal{Z}_i,$$

and therefore

$$(5.11) \quad \text{Var}(\mathcal{Z}_n) = \sum_{i,j} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j).$$

Let us define, for $i, j = 1, \dots, k$ the squares

$$(5.12) \quad S_{i,j} := I_i \times I_j = [(i-1)\delta_0, i\delta_0] \times [(j-1)\delta_0, j\delta_0],$$

so that

$$[0, L]^2 = \bigcup_{i,j} S_{i,j}.$$

Definition 3. (cf. [RW16, Definition 4.5] and [RWY16, Definition 2.7]) *We say that $S_{i,j} \subset [0, L]^2$ in (5.12) is singular if there exists $(t_1, t_2) \in S_{i,j}$ such that*

$$|r(t_1, t_2)| > 1/2.$$

For example, all *diagonal* squares $S_{i,i}$ are singular, because $r(t, t) \equiv 1$. Since $r/\sqrt{E_n}$ is a Lipschitz function with absolute constant, if $S_{i,j}$ is singular, then

$$|r(t_1, t_2)| > 1/4$$

for every $(t_1, t_2) \in S_{i,j}$, provided that c_0 is chosen sufficiently small. Upon invoking (5.11) we may write

$$(5.13) \quad \text{Var}(\mathcal{Z}_n) = \sum_{i,j:S_{i,j} \text{ non-sing.}} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) + \sum_{i,j:S_{i,j} \text{ sing.}} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j).$$

We apply the Kac-Rice formula on the non-singular squares and bound the contribution corresponding to the singular part as follows.

Lemma 5.4. *For a δ -separated sequence $\{n\} \subset S$ such that $\mathcal{N}_n \rightarrow +\infty$, we have*

$$(5.14) \quad \left| \sum_{i,j:S_{i,j} \text{ sing.}} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) \right| = o\left(\frac{n}{\mathcal{N}_n^2}\right).$$

Lemma 5.4 will be proved in §C.1. Let us now deal with the non-singular part. If $S_{i,j}$ is non-singular (Definition 3), then $r(t_1, t_2) \neq \pm 1$ for every $(t_1, t_2) \in S_{i,j}$ (note that necessarily $i \neq j$). We can apply Kac-Rice formula ([AW09], [RW16, Equality (3.2)]) to write

$$(5.15) \quad \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) = \int_{I_i} \int_{I_j} (K_2(t_1, t_2) - K_1(t_1)K_1(t_2)) dt_1 dt_2,$$

where ([RW16, Lemma 2.1])

$$K_1(t) := \phi_t(0) \cdot \mathbb{E}[f'_n(t) | f_n(t) = 0] = \sqrt{2}\sqrt{n},$$

ϕ_t being the density of the Gaussian random variable $f_n(t)$.

The function $K_2(t_1, t_2)$ is the 2-point correlation function of zeros of the process f_n (see [RW16, §3.2]), defined as follows: for $t_1 \neq t_2$

$$K_2(t_1, t_2) = \phi_{t_1, t_2}(0, 0) \cdot \mathbb{E}[|f'_n(t_1)| \cdot |f'_n(t_2)| | f_n(t_1) = f_n(t_2) = 0],$$

ϕ_{t_1, t_2} being the probability density of the Gaussian vector $(f_n(t_1), f_n(t_2))$. The function K_2 admits a continuation to a smooth function on the whole of $[0, L]^2$ (see [RW16]), though its values at the diagonal are of no significance for our purposes.

5.3.2. Proof of Lemma 5.1. First we need to Taylor expand the 2-point correlation function, which will be proven in §C.2. The following is a refinement of [RW16, Proposition 3.2], valid for any curve.

Lemma 5.5. *For every $\varepsilon > 0$, the two-point correlation function K_2 satisfies, uniformly for $|r| < 1 - \varepsilon$,*

(5.16)

$$\begin{aligned} K_2(t_1, t_2) = & \frac{\alpha}{\pi^2} \left(1 + \frac{1}{2}(r_{12}/\alpha)^2 + \frac{1}{2}r^2 - \frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{(r_1/\sqrt{\alpha})^2}{2} + \frac{3}{8}r^4 + \frac{1}{24}(r_{12}/\alpha)^4 \right. \\ & - \frac{(r_2/\sqrt{\alpha})^4}{8} - \frac{(r_1/\sqrt{\alpha})^4}{8} + (r_{12}/\alpha)r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) \\ & + \frac{(r_1/\sqrt{\alpha})^2(r_2/\sqrt{\alpha})^2}{4} - \frac{3}{4}r^2(r_2/\sqrt{\alpha})^2 - \frac{3}{4}r^2(r_1/\sqrt{\alpha})^2 + \frac{1}{4}(r_{12}/\alpha)^2r^2 \\ & + \frac{1}{4}(r_2/\sqrt{\alpha})^2(r_{12}/\alpha)^2 + \frac{1}{4}(r_1/\sqrt{\alpha})^2(r_{12}/\alpha)^2 \Big) \\ & + \alpha O \left(r^6 + (r_1/\sqrt{\alpha})^6 + (r_2/\sqrt{\alpha})^6 + (r_{12}/\alpha)^6 \right), \end{aligned}$$

where the constants involved in the “ O ”-notation depend only on ε .

We can now prove Lemma 5.1.

Proof of Lemma 5.1. Substituting (5.15) and (5.14) into (5.13), valid for non-singular sets, we may write

$$\text{Var}(\mathcal{Z}_n) = \int_{[0,L]^2 \setminus B} (K_2(t_1, t_2) - K_1(t_1)K_1(t_2)) dt_1 dt_2 + o(n\mathcal{N}_n^{-2}),$$

where B denotes the union of all singular sets $S_{i,j}$ (see (C.2)). Outside of B , we can use Lemma 5.5 together with Lemma E.2 to bound the error term. Lemma D.1 under the assumption $4B_{\mathcal{C}}(\mu) = L^2$ (and α given by (2.6)) gives

$$\int_0^L \int_0^L \left((r_{12}/\alpha)^2 + r^2 - (r_2/\sqrt{\alpha})^2 - (r_1/\sqrt{\alpha})^2 \right) dt_1 dt_2 = o(\mathcal{N}_n^{-2}),$$

and the uniform boundedness of the integrand on the r.h.s. of (5.1) allows to conclude the proof. \square

6. PROOF OF PROPOSITIONS 3.5 AND 3.6

The following lemma is (3.3) of Proposition 3.5.

Lemma 6.1. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve on the torus, of total length L . Let $\{n\} \subset S$ be a δ -separated sequence such that $\mathcal{N}_n \rightarrow +\infty$, then*

$$\text{Var}(\mathcal{Z}_n[2]) = o\left(\frac{n}{\mathcal{N}_n^2}\right).$$

Proof. First, since \mathcal{C} is static, $\mathcal{Z}_n^a[2] \equiv 0$ by (4.6). Hence we have

$$\mathcal{Z}_n[2] = \mathcal{Z}_n^b[2],$$

by Lemma 4.1. Concerning \mathcal{Z}_n^b we invoke (4.8) to bound

$$\text{Var}(\mathcal{Z}_n^b[2]) \ll \frac{n}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \frac{1}{|\lambda - \lambda'|} \ll \frac{n}{\mathcal{N}_n^2} \cdot \frac{\mathcal{N}_n^2}{n^{1/4+\delta}} = o\left(\frac{n}{\mathcal{N}_n^2}\right),$$

since our sequence of energy levels is assumed to be δ -separated (1.17). \square

In what follows we study the limiting distribution of $\mathcal{Z}_n[4]$. We have the following explicit formula for the fourth chaotic component $\mathcal{Z}_n[4]$ upon substituting $q = 2$ in (2.12):

$$(6.1) \quad \mathcal{Z}_n[4] = \sqrt{2\pi^2 n} \left(b_4 a_0 \int_0^L H_4(f_n(t)) dt + b_2 a_2 \int_0^L H_2(f_n(t)) H_2(\tilde{f}'_n(t)) dt + b_0 a_4 \int_0^L H_4(\tilde{f}'_n(t)) dt \right).$$

6.1. Preliminaries. Let us define the random variables ($n \in S$)

$$(6.2) \quad W_1(n) := \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1),$$

and the random processes

$$(6.3) \quad W_2^t(n) := \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1) 2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2.$$

on $t \in [0, L]$. In Lemma 6.3 below we will find an explicit formula for $\mathcal{Z}_n[4]$ in terms of the $W_1(n)$ and $W_2(n)$ and their by-products. To this end we first express each of the three terms in (6.1) in terms of $W_1(n)$ and $W_2(n)$ (proven in Appendix F).

Lemma 6.2. *For a δ -separated sequence $\{n\} \subset S$ such that $\mathcal{N}_n \rightarrow +\infty$, we have*

$$\begin{aligned} \int_0^L H_4(f_n(t)) dt &= X_n^a + X_n^b, \\ \int_0^L H_4(f'_n(t)) dt &= Y_n^a + Y_n^b, \\ \int_0^L H_2(f_n(t)) H_2(f'_n(t)) dt &= Z_n^a + Z_n^b, \end{aligned}$$

where for $n \in S$

$$(6.4) \quad \begin{aligned} X_n^a &:= \frac{6L}{\mathcal{N}_n} (W_1(n)^2 - 1), \\ Y_n^a &:= \frac{6L}{\mathcal{N}_n} \left(\int_0^L W_2^t(n)^2 dt - 4 \frac{1}{\mathcal{N}_n} \sum_{\lambda} \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^4 dt \right), \\ Z_n^a &:= \frac{2}{\mathcal{N}_n} \left(W_1(n) \int_0^L W_2^t(n) dt - L \right), \end{aligned}$$

and as $\mathcal{N}_n \rightarrow +\infty$

$$(6.5) \quad \text{Var}(X_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right), \text{Var}(Y_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right), \text{Var}(Z_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right).$$

(Note that the random variables X_n^a, Y_n^a, Z_n^a defined in (6.4) are centered.) We then have the following result.

Lemma 6.3. *For a δ -separated sequence $\{n\} \subset S$ such that $\mathcal{N}_n \rightarrow +\infty$, we have*

$$(6.6) \quad \mathcal{Z}_n[4] = \mathcal{Z}_n^a[4] + \mathcal{Z}_n^b[4],$$

with

$$(6.7) \quad \mathcal{Z}_n^a[4] = \frac{\sqrt{2n}}{24} (3X_n^a - Y_n^a - 6Z_n^a),$$

where X_n^a, Y_n^a and Z_n^a are as in (6.4),

$$(6.8) \quad \text{Var}(\mathcal{Z}_n^a[4]) \sim \frac{n}{4\mathcal{N}_n^2} (16A_c(\mu_n) + 24B_c(\mu_n) - 7L^2),$$

and

$$(6.9) \quad \text{Var}(\mathcal{Z}_n^b[4]) = o\left(\frac{n}{\mathcal{N}_n^2}\right).$$

Proof of Lemma 6.3 assuming Lemma 6.2. We have the explicit values

$$(6.10) \quad \begin{aligned} a_0 &= \sqrt{\frac{2}{\pi}}, & a_2 &= \sqrt{\frac{2}{\pi}} \frac{1}{2}, & a_4 &= \sqrt{\frac{2}{\pi}} \frac{-1}{2 \cdot 2^2 \cdot 3}, \\ b_0 &= \frac{1}{\sqrt{2\pi}}, & b_2 &= \frac{-1}{2\sqrt{2\pi}}, & b_4 &= \frac{3}{4!\sqrt{2\pi}} \end{aligned}$$

by (2.9) and (2.10). Substituting (6.10) into (6.1), we have

$$\begin{aligned} \mathcal{Z}_n[4] &= \frac{\sqrt{2\pi^2 n}}{\pi} \left(\frac{3}{4!} \int_0^L H_4(f_n(t)) dt - \frac{1}{4} \int_0^L H_2(f_n(t)) H_2(\tilde{f}'_n(t)) dt - \frac{1}{4!} \int_0^L H_4(\tilde{f}'_n(t)) dt \right) \\ &= \frac{\sqrt{2n}}{24} (3X_n^a - Y_n^a - 6Z_n^a + 3X_n^b - Y_n^b - 6Z_n^b), \end{aligned}$$

by Lemma 6.2; the latter is (6.6) with

$$\mathcal{Z}_n^b[4] := \frac{\sqrt{2n}}{24} (3X_n^b - Y_n^b - 6Z_n^b),$$

and (6.9) follows from (6.5).

Let us now prove (6.8). First, observe that we can write

$$(6.11) \quad \begin{aligned} 3X_n^a - Y_n^a - 6Z_n^a &= \frac{6}{\mathcal{N}_n} \left[\frac{1}{\mathcal{N}_n/2} \sum_{\lambda, \lambda' \in \Lambda_n^+} (|a_\lambda|^2 - 1)(|a_{\lambda'}|^2 - 1) \times \right. \\ &\quad \times \int_0^L \left(3 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle^2 \right) dt \\ &\quad \left. - L + 4 \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \int_0^L \langle \lambda, \dot{\gamma}(t) \rangle^4 dt \right]. \end{aligned}$$

Equality (6.11) and some straightforward computations yield

$$(6.12) \quad \text{Var}(3X_n^a - Y_n^a - 6Z_n^a) = \frac{36}{\mathcal{N}_n^2} (A_n + B_n + 24C_n),$$

where

$$\begin{aligned}
(6.13) \quad A_n &= \frac{1}{\mathcal{N}_n^2/4} \sum_{\lambda, \lambda' \in \Lambda_n^+} \left| \int_0^L \left(3 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle^2 \right) dt \right|^2, \\
B_n &= \frac{1}{\mathcal{N}_n^2/4} \sum_{\lambda, \lambda' \in \Lambda_n^+} \int_0^L \left(3 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle^2 \right) dt \\
&\quad \times \int_0^L \left(3 - 4 \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(s) \right\rangle^2 - 4 \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(s) \right\rangle^2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(s) \right\rangle^2 \right) ds, \\
C_n &= \frac{1}{\mathcal{N}_n^2/4} \sum_{\lambda \in \Lambda_n^+} \left| \int_0^L \left(3 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 - 4 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^4 \right) dt \right|^2.
\end{aligned}$$

Now, squaring out the respective terms on the r.h.s. of (6.13), with some straightforward computations we obtain

$$(6.14) \quad A_n = -9L^2 + 32B_C(\mu_n) + 16A_C(\mu_n),$$

$$(6.15) \quad B_n = -5L^2 + 16B_C(\mu_n) + 16A_C(\mu_n),$$

and that

$$(6.16) \quad C_n = O\left(\frac{1}{\mathcal{N}_n}\right).$$

Substituting (6.14), (6.15) and (6.16) into (6.12) we obtain

$$(6.17) \quad \text{Var}(3X_n^a - Y_n^a - 6Z_n^a) = \frac{36}{\mathcal{N}_n^2}(-14L^2 + 48B_C(\mu_n) + 32A_C(\mu_n) + o(1)),$$

which, in turn, yields (6.8), bearing in mind (6.7) with (6.17). \square

Proof of Proposition 3.5. First, (3.3) is the statement Lemma 6.1, and the leading term $16A_C(\mu_n) - L^2$ was shown to be bounded away from zero as part of Proposition 3.4.

We now turn to proving (3.4). Since the curve is assumed to be static, we have $4B_C(\mu_n) = L^2$. From (6.8) we have

$$\text{Var}(\mathcal{Z}_n[4]) = \frac{n}{4\mathcal{N}_n^2} (16A_C(\mu_n) - L^2) + o\left(\frac{n}{\mathcal{N}_n^2}\right),$$

which is (3.4). \square

6.2. Proof of Proposition 3.6.

6.2.1. *Auxiliary results.* Lemma 6.3 implies that for δ -separated sequences $\{n\}$ such that $\mathcal{N}_n \rightarrow +\infty$,

$$(6.18) \quad \frac{\mathcal{Z}_n[4]}{\sqrt{\text{Var}(\mathcal{Z}_n[4])}} = \frac{\mathcal{Z}_n^a[4]}{\sqrt{\text{Var}(\mathcal{Z}_n^a[4])}} + o_{\mathbb{P}}(1),$$

where $\mathcal{Z}_n^a[4]$ is defined in (6.7) and $o_{\mathbb{P}}(1)$ denotes a sequence of random variables converging to 0 in probability. We may then infer results on the limit distribution of $\mathcal{Z}_n[4]$ from the corresponding results on $\mathcal{Z}_n^a[4]$ for static curves.

Lemma 6.4. *If $4B_{\mathcal{C}}(\mu_n) - L^2 = 0$, then for $W_1(n)$ and $W_2(n)$ defined in (6.2) and (6.3) respectively we have the identity*

$$(6.19) \quad \frac{1}{L} \int_0^L W_2^t(n) dt = W_1(n).$$

Proof. The covariance matrix $\Sigma(n)$ of the Gaussian random vector $(W_1(n), \int_0^L W_2^t(n) dt)$

$$\Sigma(n) = \begin{pmatrix} 1 & L \\ L & 4B_{\mathcal{C}}(\mu_n) \end{pmatrix}$$

satisfies

$$\det \Sigma(n) = 4B_{\mathcal{C}}(\mu_n) - L^2 = 0.$$

Therefore $W_1(n)$ is a multiple of $\int_0^L W_2^t(n) dt$, which one may evaluate as given by (6.19). \square

Remark 1. Lemma 6.4 can be alternatively proved as follows. First note that for \mathcal{C} a static curve, for every $\theta \in [0, 2\pi]$, from (5.9) it holds that

$$(6.20) \quad \frac{2}{L} \int_0^L \cos^2(\phi(t) - \theta) dt = 1,$$

where $\gamma : [0, L] \rightarrow \mathbb{T}$ is a unit speed parameterization of \mathcal{C} and $\dot{\gamma}(t) = e^{i\phi(t)}$ as in (1.22). Keeping in mind the definition (6.3) of W_2 and the definition (6.2) of W_1 we write

$$\begin{aligned} \frac{1}{L} \int_0^L W_2^t(n) dt &= \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1) \frac{2}{L} \int_0^L \cos^2(\phi(t) - \theta_\lambda) dt \\ &= \frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1) = W_1(n), \end{aligned}$$

with θ_λ denoting the argument of $\lambda/|\lambda|$ viewed as a complex number ($\lambda/|\lambda| =: e^{i\theta_\lambda}$), and where we used (6.20).

Lemma 6.5. *For a static curve $\mathcal{C} \subset \mathbb{T}$ we have*

$$(6.21) \quad \begin{aligned} \mathcal{Z}_n^a[4] &= \frac{\sqrt{2n}}{4\mathcal{N}_n} \left(- \int_0^L \left(W_2^t(n) - \frac{1}{L} \int_0^L W_2^u(n) du \right)^2 dt \right. \\ &\quad \left. + 4 \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^4 dt - L \right). \end{aligned}$$

Lemma 6.5 follows directly from (6.7) and Lemma 6.4 and its proof is omitted here. In order to study the asymptotic distribution of the r.h.s. of (6.21), we need to study the asymptotic behavior of the sequence of stochastic processes $\{W_2(n)\}_n$, when $\mu_n \Rightarrow \mu$. The natural candidate to be the limiting process is the centred Gaussian process $W_2(\mu) = \{W_2^t(\mu)\}_t$ on $[0, L]$ uniquely defined by the covariance function

$$(6.22) \quad k_\mu(s, t) := \mathbb{E}[W_2^s(\mu) \cdot W_2^t(\mu)] = 4 \int_{S^1} \langle \theta, \dot{\gamma}(t) \rangle^2 \langle \theta, \dot{\gamma}(s) \rangle^2 d\mu(\theta), \quad s, t \in [0, L].$$

The kernel k_μ above is positive-definite, hence the existence of such a $W_2(\mu)$ is guaranteed by the virtue of Kolmogorov's Theorem.

Proposition 6.6. *Let $\mathcal{C} \subset \mathbb{T}$ be a static curve of length L , and $\{n\} \subset S$ a δ -separated sequence such that $\mathcal{N}_n \rightarrow +\infty$, and $\mu_n \Rightarrow \mu$. Then*

$$(6.23) \quad \frac{\mathcal{Z}_n^a[4]}{\sqrt{\text{Var}(\mathcal{Z}_n^a[4])}} \xrightarrow{d} \mathcal{I}(\mu),$$

where

$$(6.24) \quad \mathcal{I}(\mu) := \frac{-\int_0^L \left(W_2^t(\mu) - \frac{1}{L} \int_0^L W_2^u(\mu) du \right)^2 dt + \left(4 \int_{S^1} \int_0^L \langle \theta, \dot{\gamma}(t) \rangle^4 dt d\mu(\theta) - L \right)}{\sqrt{16A_C(\mu) - L^2}},$$

and $A_C(\mu)$ is as in (1.19).

Proof. Thanks to Lemma 6.5 and (6.8) for static curves, it suffices to prove that the stochastic processes $W_2(n)$ weakly converge to $W_2(\mu)$. Lindeberg's criterion (see [NP12, Remark 11.1.2]) again, and [NP12, Theorem 6.2.3] ensure that the finite dimensional distributions of $W_2(n)$ converge to those of $W_2(\mu)$, so that a standard application of Prokhorov's Theorem (see e.g. [Dud02]) allows to conclude the proof. \square

6.2.2. Proof of Proposition 3.6. Before giving a proof for Proposition 3.6, we need to introduce some more notation. Let us think of a probability measure μ on \mathcal{S}^1 as a probability measure on $[0, 2\pi]$. There exists a centered Gaussian process $\tilde{B} = \tilde{B}(\mu)$ indexed by $[0, 2\pi]$ such that

$$(6.25) \quad \text{Cov} \left(\int_0^{2\pi} 1_A(a) d\tilde{B}_a, \int_0^{2\pi} 1_B(a) d\tilde{B}_a \right) = \mu(A \cap B),$$

for any $A, B \in \mathcal{B}([0, 2\pi])$ – the Borel σ -field on the interval $[0, 2\pi]$, 1_E denoting the indicator function of the set $E \in \mathcal{B}([0, 2\pi])$.

Let us also introduce the following three centred, jointly Gaussian, random variables

$$(6.26) \quad N_1 := \int_0^{2\pi} (\cos a)^2 d\tilde{B}_a, \quad N_2 := \int_0^{2\pi} (\sin a)^2 d\tilde{B}_a, \quad N_3 := \int_0^{2\pi} \cos a \cdot \sin a d\tilde{B}_a,$$

defined as stochastic integrals on $[0, 2\pi]$ with respect to \tilde{B} .

Proof of Proposition 3.6. First, let us denote by $G(d\mu)$ a Gaussian measure on \mathcal{S}^1 with control μ (see e.g. [NP12]), i.e. a centered Gaussian family

$$G = \{G(A) : A \in \mathcal{B}(\mathcal{S}^1)\}$$

such that

$$\mathbb{E}[G(A) \cdot G(B)] = \mu(A \cap B).$$

We have the following equality in law of stochastic processes

$$(6.27) \quad W_2^t = 2 \int_{S^1} G(\mu(d\theta)) \langle \theta, \dot{\gamma}(t) \rangle^2,$$

where the l.h.s. of (6.27) is the centered Gaussian process $W_2 = W_2(\mu)$ on $[0, L]$ defined by (6.22) and the r.h.s. of (6.27) denotes the Wiener-Itô integral on the unit circle with respect to the Gaussian measure $G(d\mu)$.

From (6.27) we deduce that

$$(6.28) \quad W_2^t - \frac{1}{L} \int_0^L W_2^u du = 2 \int_{\mathcal{S}^1} G(\mu(d\theta)) \left(\langle \theta, \dot{\gamma}(t) \rangle^2 - \frac{1}{L} \int_0^L \langle \theta, \dot{\gamma}(u) \rangle^2 du \right),$$

again equality in law. We parameterize the unit circle as

$$[0, 2\pi] \ni a \mapsto (\cos a, \sin a) \in \mathcal{S}^1.$$

Recalling (6.25) and (6.26), we have by (6.28)

$$(6.29) \quad \begin{aligned} & W_2^t - \frac{1}{L} \int_0^L W_2^u du \\ & \stackrel{d}{=} 2 \int_0^{2\pi} d\tilde{B}_a \left((\cos a \cdot \dot{\gamma}_1(t) + \sin a \cdot \dot{\gamma}_2(t))^2 - \frac{1}{L} \int_0^L (\cos a \cdot \dot{\gamma}_1(u) + \sin a \cdot \dot{\gamma}_2(u))^2 du \right) \\ & = 2 \int_0^{2\pi} d\tilde{B}_a (\cos a)^2 \left(\dot{\gamma}_1(t)^2 - \frac{1}{L} \int_0^L \dot{\gamma}_1(u)^2 du \right) \\ & \quad + 2 \int_0^{2\pi} d\tilde{B}_a (\sin a)^2 \left(\dot{\gamma}_2(t)^2 - \frac{1}{L} \int_0^L \dot{\gamma}_2(u)^2 du \right) \\ & \quad + 4 \int_0^{2\pi} d\tilde{B}_a \cos a \cdot \sin a \left(\dot{\gamma}_1(t) \dot{\gamma}_2(t) - \frac{1}{L} \int_0^L \dot{\gamma}_1(u) \dot{\gamma}_2(u) du \right) \\ & = 2N_1 f(t) - 2N_2 f(t) + 4N_3 g(t), \end{aligned}$$

where f and g are as in (1.18). The covariance matrix of $N := (N_1, N_2, N_3)$ is

$$\Sigma_N := \begin{pmatrix} \frac{3+\hat{\mu}(4)}{8} & \frac{1-\hat{\mu}(4)}{8} & 0 \\ \frac{1-\hat{\mu}(4)}{8} & \frac{3+\hat{\mu}(4)}{8} & 0 \\ 0 & 0 & \frac{1-\hat{\mu}(4)}{8} \end{pmatrix}.$$

Note that N_3 is independent of N_1 and N_2 .

The eigenvalues of Σ_N are $\frac{1+\hat{\mu}(4)}{4}$, $\frac{1}{2}$ and $\frac{1-\hat{\mu}(4)}{8}$, it is then immediate that

$$(6.30) \quad \begin{pmatrix} N_1 \\ N_2 \\ N_3 \end{pmatrix} \stackrel{d}{=} \begin{pmatrix} \frac{1}{2} Z_1 + \frac{1}{\sqrt{2}} \sqrt{\frac{1+\hat{\mu}(4)}{4}} Z_2 \\ \frac{1}{2} Z_1 - \frac{1}{\sqrt{2}} \sqrt{\frac{1+\hat{\mu}(4)}{4}} Z_2 \\ \sqrt{\frac{1-\hat{\mu}(4)}{8}} Z_3 \end{pmatrix},$$

where Z_1, Z_2, Z_3 are i.i.d. standard Gaussian random variables. Substituting (6.30) into (6.29), thanks to Proposition 6.6, we can conclude the proof. \square

APPENDIX A. ON WIENER-ÎTO CHAOTIC DECOMPOSITIONS

In this section we give some background on Wiener-Îto chaotic decompositions, for more details see e.g. [NP12]. The main idea of this decomposition relies on properties of Hermite polynomials $H_k, k = 0, 1, 2, \dots$ [Sze75, §5.5]. The latter are defined recursively as follows: $H_0 \equiv 1$, and, for $k \geq 1$,

$$H_k(t) = tH_{k-1}(t) - H'_{k-1}(t), \quad t \in \mathbb{R}.$$

For instance $H_1(t) = t$, $H_2(t) = t^2 - 1$, $H_3(t) = t^3 - 3t$, $H_4(t) = t^4 - 6t^2 + 3$. The normalized sequence of Hermite polynomials

$$(A.1) \quad \{H_k/\sqrt{k!}, k \geq 0\} =: \mathbb{H}$$

is a complete orthonormal basis for the space $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), \varphi(t) dt) =: L^2(\varphi)$ of square integrable functions on the real line w.r.t. the Gaussian standard density $\varphi(t) := e^{-t^2/2}/\sqrt{2\pi}$.

The arithmetic random waves T_n defined in §1.2, and consequently their restriction to the curve $\mathcal{C} \subset \mathbb{T}$, i.e. the random process $f_n := T_n \circ \gamma$ given in (2.1), are a by-product of a family of complex Gaussian random variables $\{a_\lambda : \lambda \in \mathbb{Z}^2\}$ – defined on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ – such that

- (1) $a_\lambda = x_\lambda + iy_\lambda$, where x_λ and y_λ are two independent (real) Gaussian random variables with mean zero and variance $1/2$;
- (2) a_λ and $a_{\lambda'}$ are independent whenever $\lambda \notin \{\lambda', -\lambda'\}$;
- (3) $a_\lambda = \overline{a_{-\lambda}}$.

Let us now define the space \mathbf{A} to be the closure in $L^2(\mathbb{P})$ of all real finite linear combinations of random variables ξ of the form $\xi = z a_\lambda + \bar{z} a_{-\lambda}$, where $\lambda \in \mathbb{Z}^2$ and $z \in \mathbb{C}$. The space \mathbf{A} is a real centered Gaussian Hilbert subspace $\subseteq L^2(\mathbb{P})$.

We are now in a position to recall the definition of Wiener chaoses. Let $q \geq 0$ be an integer, the q -th Wiener chaos C_q associated with \mathbf{A} is the closure (in $L^2(\mathbb{P})$) of all real finite linear combinations of random variables of the form

$$H_{p_1}(\xi_1) \cdot H_{p_2}(\xi_2) \cdots H_{p_k}(\xi_k)$$

for $k \geq 1$, where $p_1, \dots, p_k \geq 0$ are integers satisfying

$$p_1 + \cdots + p_k = q,$$

and (ξ_1, \dots, ξ_k) is a standard real Gaussian vector extracted from \mathbf{A} (in particular $C_0 = \mathbb{R}$ since $H_0 \equiv 1$).

Taking advantage of the orthonormality and completeness properties of \mathbb{H} (A.1) in $L^2(\varphi)$, together with a standard monotone class argument (see e.g. [NP12, Theorem 2.2.4]), we can show that

$$C_q \perp C_m, \quad q \neq m,$$

where the orthogonality holds in the sense of $L^2(\mathbb{P})$, and

$$(A.2) \quad L^2(\Omega, \sigma(\mathbf{A}), \mathbb{P}) = \bigoplus_{q=0}^{\infty} C_q.$$

The orthogonal decomposition of $L^2(\Omega, \sigma(\mathbf{A}), \mathbb{P})$ in (A.2) is equivalent to the following: every square-summable real-valued functional F on \mathbf{A} can be (uniquely) represented in the form

$$(A.3) \quad F = \sum_{q=0}^{\infty} \text{proj}(F | C_q) = \sum_{q=0}^{\infty} F[q],$$

where $F[q] := \text{proj}(F | C_q)$ is the orthogonal projection onto the q -th Wiener chaos C_q , and the series converges in $L^2(\mathbb{P})$. In particular, $F[0] = \text{proj}(F | C_0) = \mathbb{E}[F]$.

In this work we are interested in the decomposition (A.3) for $F = \mathcal{Z}_n$, the nodal intersection number given in §1.2, and formally represented as an explicit functional of the underlying Gaussian field T_n in (2.3). Keeping in mind the latter formula, it is worth

noting that the random processes $f_n(t)$ and $f'_n(t) = \langle \nabla T_n(\gamma(t)), \dot{\gamma}(t) \rangle$ viewed as collections of random variables indexed by $t \in [0, L]$, are all lying in \mathbf{A} . We decide to not give all the technical details to find the chaotic expansion of \mathcal{Z}_n . For a more complete discussion on the decomposition (A.3), including all the technical details omitted in this manuscript, the interested reader might refer to [KL97]; a brief account is given in §2.2 above.

APPENDIX B. COMPUTATIONS FOR THE 2ND CHAOTIC COMPONENT

Proof of Lemma 4.1. From (2.12) with $q = 1$ we have

$$(B.1) \quad \mathcal{Z}_n[2] = \sqrt{2\pi^2 n} \left(b_2 a_0 \int_0^L H_2(f_n(t)) dt + b_0 a_2 \int_0^L H_2(\tilde{f}'_n(t)) dt \right).$$

We evaluate the first summand in the r.h.s. of (B.1) to be

$$(B.2) \quad \begin{aligned} \int_0^L H_2(f_n(t)) dt &= \int_0^L (f_n(t)^2 - 1) dt = \int_0^L (T_n(\gamma(t))^2 - 1) dt \\ &= \frac{1}{\mathcal{N}_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt - L. \end{aligned}$$

Now we are left to simplify the second summand in the right-hand side of (B.1). We get

$$(B.3) \quad \begin{aligned} \int_0^L H_2(\tilde{f}'_n(t)) dt &= \int_0^L \left((\tilde{f}'_n(t))^2 - 1 \right) dt = \frac{1}{2\pi^2 n} \int_0^L \langle \nabla T_n(\gamma(t)), \dot{\gamma}(t) \rangle^2 dt - L \\ &= \frac{1}{2\pi^2 n} \int_0^L \frac{4\pi^2}{\mathcal{N}_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda \overline{a_{\lambda'}} \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt - L \\ &= 2 \frac{1}{\mathcal{N}_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda \overline{a_{\lambda'}} \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt - L. \end{aligned}$$

Using (B.2) and (B.3) in (B.1) we obtain, taking into account that

$$b_2 a_0 = -1/(2\pi) = -b_0 a_2,$$

(B.4)

$$\begin{aligned} \mathcal{Z}_n[2] &= \sqrt{2\pi^2 n} \left(b_2 a_0 \int_0^L H_2(f_n(t)) dt + b_0 a_2 \int_0^L H_2(\tilde{f}'_n(t)) dt \right) \\ &= \frac{\sqrt{2\pi^2 n}}{2\pi} \left(-\frac{1}{\mathcal{N}_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt + L \right) \\ &\quad + \frac{\sqrt{2\pi^2 n}}{2\pi} \left(2 \frac{1}{\mathcal{N}_n} \sum_{\lambda, \lambda' \in \Lambda_n} a_\lambda \overline{a_{\lambda'}} \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt - L \right) \\ &= \frac{\sqrt{2\pi^2 n}}{2\pi} \left(-L \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} |a_\lambda|^2 + L + 2 \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} |a_\lambda|^2 \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt - L \right) \\ &\quad + \frac{\sqrt{2\pi^2 n}}{2\pi} \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L \left(2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle - 1 \right) e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt. \end{aligned}$$

Now, thanks to Lemma 4.2, we can write (B.4) as

$$\begin{aligned} \mathcal{Z}_n[2] &= \frac{\sqrt{2\pi^2 n}}{2\pi} \frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} (|a_\lambda|^2 - 1) \left(2 \int_0^L \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 dt - L \right) \\ &\quad + \frac{\sqrt{2\pi^2 n}}{2\pi} \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L \left(2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle - 1 \right) e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt, \end{aligned}$$

which equals (4.3). □

APPENDIX C. AUXILIARY RESULTS FOR THE APPROXIMATE KAC-RICE FORMULA

C.1. Contribution of singular squares. In this section we prove Lemma 5.4, following [RW16, §4]. We first need the following result, whose proof is similar to the proof of Lemma 2.4 in [RWY16] and hence omitted.

Lemma C.1. *There exists a constant $c_0 > 0$ sufficiently small such that for every $t_1, t_2 \in [0, L]$ with*

$$0 < |t_1 - t_2| < c_0 / \sqrt{E_n},$$

we have

$$r(t_1, t_2) \neq \pm 1.$$

Proposition 4.4 in [RW16] also asserts that for $t_1 \in [0, L]$ and $0 < |t_2 - t_1| < c_0 / \sqrt{n}$ one has the uniform estimate

$$(C.1) \quad K_2(t_1, t_2) = O(n).$$

Lemma C.1 and (C.1) allows to prove the following as in the proof of [RW16, Proposition 4.1].

Lemma C.2. *We have*

$$\text{Var}(\mathcal{Z}_i) = O(1),$$

uniformly for $i \leq k$, where the constants involved in the “O”-notation depend only on c_0 .

The following follows upon applying Cauchy-Schwarz with Lemma C.2.

Corollary C.3. *We have*

$$\text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) = O(1),$$

uniformly for $i, j \leq k$, where the constants involved in the “O”-notation depend only on c_0 .

Let us now denote by B the union of all singular cubes

$$(C.2) \quad B = \bigcup_{S_{i,j} \text{ singular}} S_{i,j}.$$

The proof of the following is similar to the proof of Lemma 4.7 in [RW16].

Lemma C.4. *The total area of the singular set is, for a δ -separated sequence $\{n\} \subset S$ such that $\mathcal{N}_n \rightarrow +\infty$,*

$$\text{meas}(B) = o(\mathcal{N}_n^{-2}).$$

Proof. We apply Chebyshev-Markov inequality to the measure of B to get

$$\text{meas}(B) \ll \int_0^L r(t_1, t_2)^6 dt_1 dt_2.$$

Hence bounding the measure of the singular set B is reduced to bounding the 6th moment and its derivatives. An application of Lemma D.2 below then concludes the proof of Lemma C.4. \square

We are now in a position to prove Lemma 5.4.

Proof of Lemma 5.4. Since the number of singular cubes is $O(E_n \text{meas}(B))$, Corollary C.3 bounds the contribution of singular cubes in (5.13) as

$$(C.3) \quad \left| \sum_{i,j: S_{i,j} \text{ sing.}} \text{Cov}(\mathcal{Z}_i, \mathcal{Z}_j) \right| = O(E_n \cdot \text{meas}(B));$$

The statement of Lemma 5.4 then follows upon an application of Lemma C.4. \square

C.2. Taylor expansion for the two-point correlation function.

Proof of Lemma 5.5. Recall that $\alpha = 2\pi^2 n$ as in (2.6). Lemma 3.2 in [RW16] asserts that

$$(C.4) \quad K_2 = \frac{1}{\pi^2(1-r^2)^{3/2}} \cdot \mu \cdot (\sqrt{1-\rho^2} + \rho \arcsin \rho),$$

where

$$\mu = \sqrt{\alpha(1-r^2) - r_1^2} \cdot \sqrt{\alpha(1-r^2) - r_2^2},$$

$$\rho = \frac{r_{12}(1-r^2) + rr_1r_2}{\sqrt{\alpha(1-r^2) - r_1^2} \cdot \sqrt{\alpha(1-r^2) - r_2^2}}.$$

We now set

$$G(\rho) := \frac{2}{\pi} \left(\sqrt{1-\rho^2} + \rho \arcsin \rho \right),$$

then, as $\rho \rightarrow 0$,

$$(C.5) \quad G(\rho) = \frac{2}{\pi} \left(1 + \frac{\rho^2}{2} + \frac{\rho^4}{24} + O(\rho^6) \right).$$

Let us now expand μ around 0.

$$(C.6) \quad \begin{aligned} \mu = \alpha \Big(& 1 - r^2 - \frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{(r_1/\sqrt{\alpha})^2}{2} - \frac{(r_2/\sqrt{\alpha})^4}{8} \\ & - \frac{(r_1/\sqrt{\alpha})^4}{8} + \frac{(r_1/\sqrt{\alpha})^2(r_2/\sqrt{\alpha})^2}{4} + O((r^2 + (r_2/\sqrt{\alpha})^2)^3) \Big). \end{aligned}$$

Moreover,

$$(C.7) \quad \frac{1}{(1-r^2)^{3/2}} = 1 + \frac{3}{2}r^2 + \frac{15}{8}r^4 + O(r^6).$$

Let us now Taylor expand ρ .

$$(C.8) \quad \begin{aligned} \rho = & (r_{12}/\alpha) + (r_{12}/\alpha) \frac{(r_2/\sqrt{\alpha})^2}{2} + (r_{12}/\alpha) \frac{(r_1/\sqrt{\alpha})^2}{2} + r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) \\ & + O(r_{12}(r^2 + (r_2/\sqrt{\alpha})^2)^2). \end{aligned}$$

Substituting (C.8) into (C.5) we obtain

$$(C.9) \quad \begin{aligned} G(\rho) = & \frac{2}{\pi} \left(1 + \frac{1}{2} ((r_{12}/\alpha)^2 + (r_{12}/\alpha)^2 (r_2/\sqrt{\alpha})^2) + (r_{12}/\alpha)^2 (r_1/\sqrt{\alpha})^2 \right) \\ & + 2(r_{12}/\alpha) r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) + \frac{1}{24} (r_{12}/\alpha)^4 + O(r_{12}^2(r^2 + (r_2/\sqrt{\alpha})^2)^2). \end{aligned}$$

Finally, using (C.7), (C.6) and (C.9) in (C.4) we get

$$\begin{aligned} K = & \frac{\alpha}{\pi^2} \left(1 + \frac{1}{2} (r_{12}/\alpha)^2 + \frac{1}{2} r^2 - \frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{(r_1/\sqrt{\alpha})^2}{2} \right. \\ & + \frac{3}{8} r^4 + \frac{1}{24} (r_{12}/\alpha)^4 - \frac{(r_2/\sqrt{\alpha})^4}{8} - \frac{(r_1/\sqrt{\alpha})^4}{8} \\ & + (r_{12}/\alpha) r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) + \frac{(r_1/\sqrt{\alpha})^2 (r_2/\sqrt{\alpha})^2}{4} \\ & - \frac{3}{2} r^2 \frac{(r_2/\sqrt{\alpha})^2}{2} - \frac{3}{2} r^2 \frac{(r_1/\sqrt{\alpha})^2}{2} + \frac{1}{2} (r_{12}/\alpha)^2 \frac{1}{2} r^2 \\ & + \frac{1}{4} (r_2/\sqrt{\alpha})^2 (r_{12}/\alpha)^2 + \frac{1}{4} (r_1/\sqrt{\alpha})^2 (r_{12}/\alpha)^2 \\ & \left. + O(r^6 + (r_1/\sqrt{\alpha})^6 + (r_2/\sqrt{\alpha})^6 + (r_{12}/\alpha)^6) \right), \end{aligned}$$

which is (5.16). □

APPENDIX D. MOMENTS OF r AND ITS DERIVATIVES

Lemma D.1. *If $\mathcal{C} \subset \mathbb{T}$ is a smooth curve with nowhere vanishing curvature, then for a δ -separated sequence $\{n\}$ such that $\mathcal{N}_n \rightarrow +\infty$, we have*

$$\begin{aligned} 1) \quad & \int_0^L \int_0^L r(t_1, t_2)^2 dt_1 dt_2 = \frac{L^2}{\mathcal{N}_n} + o\left(\frac{1}{\mathcal{N}_n^2}\right), \\ 2) \quad & \int_0^L \int_0^L \left| \frac{1}{\sqrt{4\pi^2 n}} r_1(t_1, t_2) \right|^2 dt_1 dt_2 = \frac{L^2}{2\mathcal{N}_n} + o\left(\frac{1}{\mathcal{N}_n^2}\right), \\ 3) \quad & \int_0^L \int_0^L \left| \frac{1}{4\pi^2 n} r_{12}(t_1, t_2) \right|^2 dt_1 dt_2 = \frac{B_{\mathcal{C}}(\mu_n)}{\mathcal{N}_n} + o\left(\frac{1}{\mathcal{N}_n^2}\right), \end{aligned}$$

where $B_{\mathcal{C}}(\mu_n)$ is given in (1.11).

Proof. Let us start with 1). Squaring out, we have (on separating the diagonal $\lambda = \lambda'$)

$$(D.1) \quad \int_0^L \int_0^L r(t_1, t_2)^2 dt_1 dt_2 = \frac{L^2}{\mathcal{N}_n} + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \left| \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \right|^2.$$

Lemma 5.2 in [RW16] yields

$$\int_0^L e^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle} dt \ll \frac{1}{|\lambda-\lambda'|^{1/2}},$$

therefore the contribution of the off-diagonal pairs in (D.1) is

$$\frac{1}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \left| \int_0^L e^{i2\pi\langle\lambda-\lambda',\gamma(t)\rangle} dt \right|^2 \ll \frac{1}{\mathcal{N}_n^2} \sum_{\lambda \neq \lambda'} \frac{1}{|\lambda-\lambda'|}.$$

Condition (1.17) then allows to conclude part 1). The remaining terms can be dealt in a similar way to the proof of [RW16, Proposition 5.1], taking into account (1.17) to control the contribution of the “off-diagonal terms”. \square

Proof of Lemma 5.3. Let us prove 1). We can write

$$\begin{aligned} (D.2) \quad \int_{\mathcal{C}} \int_{\mathcal{C}} r(t_1, t_2)^4 dt_1 dt_2 &= \int_{\mathcal{C}} \int_{\mathcal{C}} \left(\frac{1}{\mathcal{N}_n} \sum_{\lambda} e^{i2\pi\langle\lambda, \gamma(t_1) - \gamma(t_2)\rangle} \right)^4 dt_1 dt_2 \\ &= \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1, \dots, \lambda_4} e^{i2\pi\langle\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \gamma(t_1) - \gamma(t_2)\rangle} dt_1 dt_2 \end{aligned}$$

Now let us split the summation on the r.h.s. of (D.2) into two sums: one over quadruples $(\lambda_1, \dots, \lambda_4)$ such that $\lambda_1 + \dots + \lambda_4 = 0$ and the other one over quadruples $(\lambda_1, \dots, \lambda_4)$ such that $\lambda_1 + \dots + \lambda_4 \neq 0$:

$$\begin{aligned} (D.3) \quad \int_{\mathcal{C}} \int_{\mathcal{C}} r(t_1, t_2)^4 dt_1 dt_2 &= L^2 \frac{|S_4(n)|}{\mathcal{N}_n^4} + \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \neq 0} e^{i2\pi\langle\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \gamma(t_1) - \gamma(t_2)\rangle} dt_1 dt_2 \\ &= L^2 \frac{|S_4(n)|}{\mathcal{N}_n^4} + \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \neq 0} \left| \int_{\mathcal{C}} e^{i2\pi\langle\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \gamma(t_1)\rangle} dt_1 \right|^2, \end{aligned}$$

where

$$S_4(n) := \{(\lambda_1, \dots, \lambda_4) \in \Lambda_n^4 : \lambda_1 + \dots + \lambda_4 = 0\}.$$

Recall that [KKW13]

$$(D.4) \quad |S_4(n)| = 3\mathcal{N}_n(\mathcal{N}_n - 1),$$

and moreover [RW16, Lemma 5.2]

$$(D.5) \quad \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \neq 0} \left| \int_0^L e^{i2\pi\langle\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \gamma(t_1)\rangle} dt_1 \right|^2 \ll \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \neq 0} \frac{1}{|\lambda_1 + \dots + \lambda_4|}.$$

Substituting (D.4) and (D.5) into (D.3) we get

$$(D.6) \quad \int_{\mathcal{C}} \int_{\mathcal{C}} r(t_1, t_2)^4 dt_1 dt_2 = 3L^2 \frac{1}{\mathcal{N}_n^2} + O\left(\frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4 \neq 0} \frac{1}{|\lambda_1 + \dots + \lambda_4|}\right).$$

Lemma 5.2, in particular (2.5), then allows to estimate the error on the r.h.s. of (D.6), thus concluding the proof of 1).

Let us now deal with 4).

(D.7)

$$\begin{aligned}
& \int_{\mathcal{C}} \int_{\mathcal{C}} (r_{12}/\alpha) r(r_1/\sqrt{\alpha})(r_2/\sqrt{\alpha}) dt_1 dt_2 \\
&= 4 \cdot \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1, \dots, \lambda_4} \left\langle \frac{\lambda_1}{|\lambda_1|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_1}{|\lambda_1|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda_3}{|\lambda_3|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_4}{|\lambda_4|}, \dot{\gamma}(t_2) \right\rangle \times \\
&\quad \times e^{i2\pi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \gamma(t_1) - \gamma(t_2))} dt_1 dt_2 \\
&= -4 \cdot \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda, \lambda'} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_2) \right\rangle dt_1 dt_2 \\
&\quad - 8 \cdot \underbrace{\int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda, \lambda'} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_1) \right\rangle^2 \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_2) \right\rangle dt_1 dt_2}_{=0} \\
&\quad + 4 \cdot \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 + \dots + \lambda_4 \neq 0} \left\langle \frac{\lambda_1}{|\lambda_1|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_1}{|\lambda_1|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda_3}{|\lambda_3|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_4}{|\lambda_4|}, \dot{\gamma}(t_2) \right\rangle \times \\
&\quad \times e^{i2\pi(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4, \gamma(t_1) - \gamma(t_2))} dt_1 dt_2 \\
&= -4F_{\mathcal{C}}(\mu_n) \frac{1}{\mathcal{N}_n^2} + o(\mathcal{N}_n^{-2}),
\end{aligned}$$

where in the last step we used (1.18), and Cauchy-Schwarz inequality and again (5.2) to bound the contribution of “off-diagonal” terms. Let us now study 5).

(D.8)

$$\begin{aligned}
& \int_{\mathcal{C}} \int_{\mathcal{C}} (r_1/\sqrt{\alpha})^2 (r_2/\sqrt{\alpha})^2 dt_1 dt_2 \\
&= 4 \cdot \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1, \dots, \lambda_4} \left\langle \frac{\lambda_1}{|\lambda_1|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_2}{|\lambda_2|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_3}{|\lambda_3|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda_4}{|\lambda_4|}, \dot{\gamma}(t_2) \right\rangle \times \\
&\quad \times e^{i2\pi(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \gamma(t_1) - \gamma(t_2))} dt_1 dt_2 \\
&= 4 \cdot \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda, \lambda'} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_1) \right\rangle^2 \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_2) \right\rangle^2 dt_1 dt_2 \\
&\quad + 4 \cdot 2 \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda, \lambda'} \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t_2) \right\rangle dt_1 dt_2 \\
&\quad + 4 \cdot \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 + \dots + \lambda_4 \neq 0} \left\langle \frac{\lambda_1}{|\lambda_1|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_2}{|\lambda_2|}, \dot{\gamma}(t_1) \right\rangle \left\langle \frac{\lambda_3}{|\lambda_3|}, \dot{\gamma}(t_2) \right\rangle \left\langle \frac{\lambda_4}{|\lambda_4|}, \dot{\gamma}(t_2) \right\rangle \times \\
&\quad \times e^{i2\pi(\lambda_1 - \lambda_2 + \lambda_3 - \lambda_4, \gamma(t_1) - \gamma(t_2))} dt_1 dt_2 \\
&= L^2 \frac{1}{\mathcal{N}_n^2} + 4 \cdot 2 \frac{1}{\mathcal{N}_n^2} F_n + o(\mathcal{N}_n^{-2}),
\end{aligned}$$

where for the last step we used the well-known equality $\frac{1}{\mathcal{N}_n} \sum_{\lambda} \left\langle \frac{\lambda}{|\lambda|}, v \right\rangle^2 = \frac{1}{2}$ which holds for every unit vector v , and still Cauchy-Schwarz inequality and then (5.2) to bound the contribution of “off-diagonal” terms.

The proof of 2) is analogous to that of 1), whereas the proofs of 3), 6)-8) are analogous to that of 5) above, and hence omitted. \square

Lemma D.2. *If $\mathcal{C} \subset \mathbb{T}$ is a smooth curve with nowhere vanishing curvature, then for δ -separated sequences $\{n\}$ such that $\mathcal{N}_n \rightarrow +\infty$, we have*

$$(D.9) \quad \begin{aligned} 1) \quad & \int_{\mathcal{C}} \int_{\mathcal{C}} r(t_1, t_2)^6 dt_1 dt_2 = o(\mathcal{N}_n^{-2}), \\ 2) \quad & \int_{\mathcal{C}} \int_{\mathcal{C}} (r_1/\sqrt{\alpha})^6 dt_1 dt_2 = o(\mathcal{N}_n^{-2}), \\ 3) \quad & \int_{\mathcal{C}} \int_{\mathcal{C}} (r_{12}/\alpha)^6 dt_1 dt_2 = o(\mathcal{N}_n^{-2}). \end{aligned}$$

Proof. Let us prove 1).

$$(D.10) \quad \begin{aligned} \int_{\mathcal{C}} \int_{\mathcal{C}} r(t_1, t_2)^6 dt_1 dt_2 &= \int_{\mathcal{C}} \int_{\mathcal{C}} \left(\frac{1}{\mathcal{N}_n} \sum_{\lambda \in \Lambda_n} e_{\lambda}(\gamma(t_1) - \gamma(t_2)) \right)^6 dt_1 dt_2 \\ &= \int_{\mathcal{C}} \int_{\mathcal{C}} \frac{1}{\mathcal{N}_n^6} \sum_{\lambda_1, \dots, \lambda_6 \in \Lambda_n} e_{\lambda_1 + \dots + \lambda_6}(\gamma(t_1) - \gamma(t_2)) dt_1 dt_2 \\ &= \frac{1}{\mathcal{N}_n^6} \sum_{\lambda_1, \dots, \lambda_6 \in \Lambda_n} \left| \int_{\mathcal{C}} e_{\lambda_1 + \dots + \lambda_6}(\gamma(t)) dt \right|^2 \\ &= L^2 \frac{|S_6(n)|}{\mathcal{N}_n^6} + \frac{1}{\mathcal{N}_n^6} \sum_{\lambda_1 + \dots + \lambda_6 \neq 0} \left| \int_{\mathcal{C}} e_{\lambda_1 + \dots + \lambda_6}(\gamma(t)) dt \right|^2 \\ &\ll \frac{|S_6(n)|}{\mathcal{N}_n^6} + \frac{1}{\mathcal{N}_n^6} \sum_{\lambda_1 + \dots + \lambda_6 \neq 0} \frac{1}{|\lambda_1 + \dots + \lambda_6|}, \end{aligned}$$

where

$$S_6(n) := \{(\lambda_1, \dots, \lambda_6) \in \Lambda_n^6 : \lambda_1 + \dots + \lambda_6 = 0\},$$

and for the last step we used Lemma 5.2 in [RW16]. Recall now that [BB15]

$$|S_6(n)| = O(\mathcal{N}_n^{7/2}).$$

Using the latter together with Lemma E.2 (just below) in (D.10), we can conclude the proof of 1). The proofs of the remaining cases 2), 3) are similar to that of 1) and hence omitted. \square

APPENDIX E. CONTRIBUTION OF “OFF-DIAGONAL” TERMS

E.1. Proof of Lemma 5.2. Let us start with the following simple lemma.

Lemma E.1. *Let A, B, C, D be four points on \mathbb{S}^1 such that the segment AC intersects the segment BD , and O be the centre of the circle. Then the angle between AC and BD equals*

$$\frac{\angle AOB + \angle COD}{2}.$$

Proof. Let E be the intersection point of AC with BD and α the corresponding angle. Then, as α is the external angle in the triangle AED we have that

$$(E.1) \quad \alpha = \angle ADE + \angle EAD = \frac{\angle AOB + \angle COD}{2},$$

since for any chord XY on the circle, the angle $\angle XOY$ is twice the angle $\angle XZY$ subtended by another point Z on the circle. □

We are now in a position to prove Lemma 5.2.

Proof of Lemma 5.2. First suppose that the segment $\lambda_1\lambda_3$ intersects the segment $\lambda_2\lambda_4$. For $u = \lambda_3 - \lambda_1$ and $w = \lambda_4 - \lambda_2$, $v := u - w$, and α the angle between u and w we have

$$\begin{aligned} \|v\|^2 &= \|u\|^2 + \|w\|^2 - 2\|u\| \cdot \|w\| \cos \alpha = (\|u\| - \|w\|)^2 + 2\|u\| \cdot \|w\|(1 - \cos \alpha) \\ &\geq 2\|u\|\|w\|(1 - \cos \alpha) \gg 2\|u\| \cdot \|w\| \cdot \alpha^2, \end{aligned}$$

provided that $\alpha \in [0, \pi]$ (say). Since $\|u\| \cdot \|w\| \gg n^{1/2+2\delta}$ by assumption (1.17), and

$$\alpha \gg n^{-1/4+\delta}$$

by Lemma E.1, we have

$$\|v\| \gg n^{1+2\delta} \cdot n^{-1/2+2\delta} = n^{4\delta},$$

which, in case $\lambda_1\lambda_3$ intersects $\lambda_2\lambda_4$, is stronger than claimed.

Otherwise, let us assume that $\lambda_1\lambda_3$ does not intersect $\lambda_2\lambda_4$. That means that both λ_1, λ_3 are lying on the same arc $\widehat{\lambda_2\lambda_4}$ (one of two choices) and either the arcs $\widehat{\lambda_2\lambda_1}$, $\widehat{\lambda_1\lambda_3}$, $\widehat{\lambda_3\lambda_4}$ are pairwise disjoint or the arcs $\widehat{\lambda_4\lambda_1}$, $\widehat{\lambda_1\lambda_3}$ and $\widehat{\lambda_3\lambda_2}$ are pairwise disjoint; we assume w.l.o.g that the former holds. Since

$$\lambda_3 - \lambda_1 = (-\lambda_1) - (-\lambda_3),$$

and upon replacing (λ_1, λ_3) by $(-\lambda_3, -\lambda_1)$ if necessary, we may assume that λ_1, λ_3 are lying in the smaller of the arcs $\widehat{\lambda_2\lambda_4}$, i.e. the angle $\lambda_2 O \lambda_4 < \pi$.

Denote as before $u = \lambda_3 - \lambda_1$ and $w = \lambda_4 - \lambda_2$, $v := u - w$. Here we claim that $\|u\| - \|w\|$ is not too small, and then, by the triangle inequality, so is

$$\|v\| \geq \left| \|u\| - \|w\| \right|.$$

Let α be the angle $\alpha = \angle \lambda_2 O \lambda_4$, and the angle $\beta = \angle \lambda_1 O \lambda_3 < \alpha$. We have $\|u\| = 2\sqrt{n} \sin(\alpha/2)$, $\|w\| = 2\sqrt{n} \sin(\beta/2)$, so that

$$\|u\| - \|w\| = 2\sqrt{n} \sin\left(\frac{\alpha - \beta}{4}\right) \cos\left(\frac{\alpha + \beta}{4}\right).$$

However, by the assumption (1.17) we have that both $\alpha - \beta \gg n^{-1/4+\delta}$ and $2\pi - (\alpha + \beta) \gg n^{-1/4+\delta}$ as at least one of α or β falls short of π by at least $\gg n^{-1/4+\delta}$. Hence

$$\|u\| - \|w\| \gg \sqrt{n} \cdot n^{-1/4+\delta} \cdot n^{-1/4+\delta} = n^{2\delta}$$

which yields the statement of Lemma 5.2 in this case. \square

E.2. Contribution of the higher order chaoses.

Lemma E.2. *For δ -separated sequences $\{n\} \subset S$ we have the bound*

$$(E.2) \quad \frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 + \dots + \lambda_6 \neq 0} \frac{1}{\|\lambda_1 + \dots + \lambda_6\|} = o_{\mathcal{N}_n \rightarrow \infty}(1).$$

Proof. Let $\epsilon > 0$ be positive number, and set $A := \frac{1}{2}\mathcal{N}_n^{2+\epsilon}$ (say). We distinguish between two cases:

i) $\|\lambda_1 + \dots + \lambda_6\| \geq A$, ii) $\|\lambda_1 + \dots + \lambda_6\| < A$. The contribution of all summands with i) holding is

$$\ll \frac{1}{A} \ll \frac{1}{\mathcal{N}_n^\epsilon},$$

hence we are only to bound the contribution of summands satisfying ii).

Assume here that ii) indeed holds. We claim that for fixed $\lambda_1, \dots, \lambda_4$ such that $\lambda_1 + \dots + \lambda_4 \neq 0$, there exist $O(1)$ -choices for λ_5, λ_6 . Indeed, if we assume that both $\|\lambda_1 + \dots + \lambda_4 + \lambda_5 + \lambda_6\| < A$ and $\|\lambda_1 + \dots + \lambda_4 + \lambda'_5 + \lambda'_6\| < A$, then, by the triangle inequality, we obtain

$$(E.3) \quad \|\lambda_5 + \lambda_6 - \lambda'_5 - \lambda'_6\| < 2A.$$

By Lemma 5.2 and in light of (1.6), (E.3) is valid only if $\lambda_5 + \lambda_6 - \lambda'_5 - \lambda'_6 = 0$. This in turn implies that either $\lambda_5 = -\lambda_6$ or $\lambda_5 = \lambda'_5$ or $\lambda_5 = \lambda'_6$. The possibility $\lambda_5 = -\lambda_6$ is not valid, since then $\lambda_1 + \dots + \lambda_6 = \lambda_1 + \dots + \lambda_4$, and therefore given $\lambda_1, \dots, \lambda_4$ there could be at most two choices for λ_5, λ_6 such that ii) holds.

Let $B > 0$ be a large (but fixed) parameter, and assume that ii) holds. We distinguish between further two cases: a) $\|\lambda_1 + \dots + \lambda_6\| \geq B$, or b) $\|\lambda_1 + \dots + \lambda_6\| < B$. By the above, the contribution of summands satisfying a) to the l.h.s. of (E.2) is $O(\frac{1}{B})$.

To treat terms that satisfy b) we recall that [KKW13, Theorem 2.2] shows that the number of 6-tuples $(\lambda_1, \dots, \lambda_6) \in \Lambda_n^6$ satisfying $\lambda_1 + \dots + \lambda_6 = 0$ is $o(\mathcal{N}_n^4)$. A slight modification⁴ of the proof of [KKW13, Theorem 2.2] yields that the same holds for an arbitrary *fixed* $v \in \mathbb{Z}^2$ in place of 0, i.e. the number of tuples such 6-tuples satisfying

$$\lambda_1 + \dots + \lambda_6 = v$$

is $o(\mathcal{N}_n^4)$. Therefore the total contribution to (E.2) of summands satisfying b) is $o_B(1)$. Consolidating the various bounds we encountered, we have that

$$\frac{1}{\mathcal{N}_n^4} \sum_{\lambda_1 + \dots + \lambda_6 \neq 0} \frac{1}{\|\lambda_1 + \dots + \lambda_6\|} \ll \frac{1}{\mathcal{N}_n^\epsilon} + \frac{1}{B} + o_B(1),$$

which certainly implies (E.2). \square

⁴In Eq. 59, consider $y_1 + y_2 \in A + v$ and in Eq. 60, consider $y_1 + y_2 \in A + v$ so that we have $\langle 1_A \star 1_A, 1_{A+v} \rangle$

APPENDIX F. AUXILIARY COMPUTATIONS FOR THE FOURTH CHAOS

Proof of Lemma 6.2. Let us start with 1). Recall the expression for the fourth Hermite polynomial: $H_4(t) = t^4 - 6t^2 + 3$.

$$\begin{aligned}
 (F.1) \quad 1) \int_0^L H_4(f_n(t)) dt &= \int_0^L (f_n(t)^4 - 6f_n(t)^2 + 3) dt \\
 &= \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda'''} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt \\
 &\quad - 6 \frac{1}{\mathcal{N}_n} \sum_{\lambda, \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt + 3L.
 \end{aligned}$$

Let us divide the first term on the r.h.s. of (F.1) into two summation, one over the quadruples in $S_4(n)$ and the other one over the quadruples not belonging to $S_4(n)$.

$$\begin{aligned}
 (F.2) \quad \int_0^L H_4(f_n(t)) dt &= 3L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda'} |a_\lambda|^2 |a_{\lambda'}|^2 - 3L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 \\
 &\quad + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt \\
 &\quad - 6 \frac{1}{\mathcal{N}_n} \sum_{\lambda, \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt + 3L.
 \end{aligned}$$

Now let us divide the fourth term on the r.h.s. of (F.2) into two sums, one for the diagonal terms and the other one for the off-diagonal terms. That is,

$$\begin{aligned}
 (F.3) \quad \int_0^L H_4(f_n(t)) dt &= 3L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda'} |a_\lambda|^2 |a_{\lambda'}|^2 - 6L \frac{1}{\mathcal{N}_n} \sum_{\lambda} |a_\lambda|^2 + 3L \\
 &\quad - 6 \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt - 3L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 \\
 &\quad + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt.
 \end{aligned}$$

Note that the first three terms on the r.h.s. of (F.3) can be rewritten as

$$\begin{aligned}
\int_0^L H_4(f_n(t)) dt &= 3 \cdot 2 \cdot L \frac{1}{\mathcal{N}_n} \left(\frac{1}{\sqrt{\mathcal{N}_n/2}} \sum_{\lambda \in \Lambda_n^+} (|a_\lambda|^2 - 1) \right)^2 \\
&\quad - 3L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 - 6 \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \\
&\quad + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt \\
&= 6 \cdot L \frac{1}{\mathcal{N}_n} W_1(n)^2 - 3L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 - 6 \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \\
&\quad + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt \\
&= X_n^a + X_n^b,
\end{aligned}$$

where $W_1(n)$ is defined as in (6.2), X_n^a is given in (6.4) and

$$\begin{aligned}
(F.4) \quad X_n^b &:= -3L \frac{1}{\mathcal{N}_n^2} \sum_{\lambda} (|a_\lambda|^4 - 2) - 6 \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \\
&\quad + \frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt.
\end{aligned}$$

Let us prove that, as $\mathcal{N}_n \rightarrow +\infty$,

$$(F.5) \quad \text{Var}(X_n^b) = o\left(\frac{1}{\mathcal{N}_n^2}\right).$$

For the first term on the r.h.s. of (F.4) we have

$$(F.6) \quad \text{Var}\left(\frac{1}{\mathcal{N}_n^2} \sum_{\lambda} (|a_\lambda|^4 - 2)\right) = O(\mathcal{N}_n^{-3})$$

whereas for the second one [RW16, (5.18)] and (1.17) give

$$(F.7) \quad \text{Var}\left(\frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt\right) = o\left(\frac{1}{\mathcal{N}_n^2}\right).$$

For the last term

$$(F.8) \quad \text{Var}\left(\frac{1}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt\right) = o\left(\frac{1}{\mathcal{N}_n^2}\right).$$

follows from [RW16, (5.18)] and Lemma 5.2. This concludes the proof of 1).

Let us now study the second summand in (6.1). The argument given below is similar to the one above concerning the first summand in (6.1); accordingly we will omit some

technical details.

$$\begin{aligned}
 (F.9) \quad \int_0^L H_4(f'_n(t)) dt &= \int_0^L (f'_n(t)^4 - 6f'_n(t)^2 + 3) dt \\
 &= \frac{1}{(2\pi^2 n)^2} \int_0^L \frac{(2\pi)^4}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda'''} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \times \\
 &\quad \times \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle \langle \lambda'', \dot{\gamma}(t) \rangle \langle \lambda''', \dot{\gamma}(t) \rangle e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt \\
 &\quad - 6 \frac{1}{2\pi^2 n} \int_0^L \frac{(2\pi)^2}{\mathcal{N}_n} \sum_{\lambda, \lambda'} a_\lambda \overline{a_{\lambda'}} \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt + 3L.
 \end{aligned}$$

Dealing with (F.9) as in (F.3) we obtain

$$\begin{aligned}
 \int_0^L H_4(f'_n(t)) dt &= 3 \int_0^L \left(2 \frac{1}{\mathcal{N}_n} \sum_{\lambda} (|a_\lambda|^2 - 1) \left\langle \frac{\lambda}{|\lambda|}, \dot{\gamma}(t) \right\rangle^2 \right)^2 dt \\
 &\quad - 3 \frac{1}{(2\pi^2 n)^2} \frac{(2\pi)^4}{\mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 \int_0^L \langle \lambda, \dot{\gamma}(t) \rangle^4 dt \\
 &\quad + \frac{1}{(2\pi^2 n)^2} \int_0^L \frac{(2\pi)^4}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \times \\
 &\quad \times \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle \langle \lambda'', \dot{\gamma}(t) \rangle \langle \lambda''', \dot{\gamma}(t) \rangle e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt \\
 &\quad - 6 \frac{1}{2\pi^2 n} \int_0^L \frac{(2\pi)^2}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt \\
 &= Y_n^a + Y_n^b,
 \end{aligned}$$

where Y_n^a is defined as in (6.4) and

$$\begin{aligned}
 Y_n^b &:= -3 \frac{1}{(2\pi^2 n)^2} \frac{(2\pi)^4}{\mathcal{N}_n^2} \sum_{\lambda} (|a_\lambda|^4 - 2) \int_0^L \langle \lambda, \dot{\gamma}(t) \rangle^4 dt \\
 &\quad + \frac{1}{(2\pi^2 n)^2} \int_0^L \frac{(2\pi)^4}{\mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \times \\
 &\quad \times \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle \langle \lambda'', \dot{\gamma}(t) \rangle \langle \lambda''', \dot{\gamma}(t) \rangle e^{i2\pi \langle \lambda - \lambda' + \lambda'' - \lambda''', \gamma(t) \rangle} dt \\
 &\quad - 6 \frac{1}{2\pi^2 n} \int_0^L \frac{(2\pi)^2}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle e^{i2\pi \langle \lambda - \lambda', \gamma(t) \rangle} dt.
 \end{aligned}$$

We are now left with the third summand in (6.1).

(F.10)

$$\begin{aligned}
\int_0^L H_2(f_n(t)) H_2(f'_n(t)) dt &= \int_0^L (f_n(t)^2 - 1)(f'_n(t)^2 - 1) dt \\
&= \frac{(2\pi)^2}{2\pi^2 n \mathcal{N}_n^2} \int_0^L \sum_{\lambda, \lambda', \lambda'', \lambda'''} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \times \\
&\quad \times \langle \lambda'', \dot{\gamma}(t) \rangle \langle \lambda''', \dot{\gamma}(t) \rangle e^{i2\pi(\lambda - \lambda' + \lambda'' - \lambda''', \gamma(t))} dt \\
&\quad - \frac{1}{\mathcal{N}_n} \sum_{\lambda, \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi(\lambda - \lambda', \gamma(t))} dt \\
&\quad - \frac{1}{2\pi^2 n} \int_0^L \frac{(2\pi)^2}{\mathcal{N}_n} \sum_{\lambda, \lambda'} a_\lambda \overline{a_{\lambda'}} \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle e^{i2\pi(\lambda - \lambda', \gamma(t))} dt + L
\end{aligned}$$

Finally, we can rewrite (F.10) as

$$\begin{aligned}
&\int_0^L H_2(f_n(t)) H_2(f'_n(t)) dt \\
&= \left(\frac{1}{\mathcal{N}_n} \sum_{\lambda} (|a_\lambda|^2 - 1) \right) \left(\frac{1}{\mathcal{N}_n} \sum_{\lambda'} (|a'_{\lambda}|^2 - 1) \int_0^L 2 \left\langle \frac{\lambda'}{|\lambda'|}, \dot{\gamma}(t) \right\rangle^2 dt \right) \\
&\quad - \frac{(2\pi)^2}{2\pi^2 n \mathcal{N}_n^2} \sum_{\lambda} |a_\lambda|^4 \int_0^L \langle \lambda, \dot{\gamma}(t) \rangle^2 dt \\
&\quad + \frac{(2\pi)^2}{2\pi^2 n \mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L \langle \lambda'', \dot{\gamma}(t) \rangle \langle \lambda''', \dot{\gamma}(t) \rangle \times \\
&\quad \times e^{i2\pi(\lambda - \lambda' + \lambda'' - \lambda''', \gamma(t))} dt - \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi(\lambda - \lambda', \gamma(t))} dt \\
&\quad - \frac{1}{2\pi^2 n} \int_0^L \frac{(2\pi)^2}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle e^{i2\pi(\lambda - \lambda', \gamma(t))} dt \\
&= Z_n^a + Z_n^b,
\end{aligned}$$

where Z_n^a is defined as in (6.4) and

$$\begin{aligned}
Z_n^b &:= - \frac{(2\pi)^2}{2\pi^2 n \mathcal{N}_n^2} \sum_{\lambda} (|a_\lambda|^4 - 2) \int_0^L \langle \lambda, \dot{\gamma}(t) \rangle^2 dt \\
&\quad + \frac{(2\pi)^2}{2\pi^2 n \mathcal{N}_n^2} \sum_{\lambda, \lambda', \lambda'', \lambda''' \notin S_n(4)} a_\lambda \overline{a_{\lambda'}} a_{\lambda''} \overline{a_{\lambda'''}} \int_0^L \langle \lambda'', \dot{\gamma}(t) \rangle \langle \lambda''', \dot{\gamma}(t) \rangle e^{i2\pi(\lambda - \lambda' + \lambda'' - \lambda''', \gamma(t))} dt \\
&\quad - \frac{1}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \int_0^L e^{i2\pi(\lambda - \lambda', \gamma(t))} dt \\
&\quad - \frac{1}{2\pi^2 n} \int_0^L \frac{(2\pi)^2}{\mathcal{N}_n} \sum_{\lambda \neq \lambda'} a_\lambda \overline{a_{\lambda'}} \langle \lambda, \dot{\gamma}(t) \rangle \langle \lambda', \dot{\gamma}(t) \rangle e^{i2\pi(\lambda - \lambda', \gamma(t))} dt.
\end{aligned}$$

□

APPENDIX G. A FAMILY OF STATIC CURVES

Proposition G.1. *Let $\mathcal{C} \subset \mathbb{T}$ be a smooth closed curve with nowhere vanishing curvature, invariant w.r.t. rotation by $2\pi/k$ for some $k \geq 3$. Then \mathcal{C} is static.*

Proof. To show that \mathcal{C} is static it is sufficient [RW16, Corollary 7.2] to see that

$$(G.1) \quad B_{\mathcal{C}} \left(\frac{d\theta}{2\pi} \right) = \frac{L^2}{4},$$

i.e. check the condition (1.16) with $\mu = \frac{d\theta}{2\pi}$ only. By inverting the order of integration in (1.11) we have that (G.1) is equivalent to

$$(G.2) \quad \int_0^{2\pi} \left(\int_0^L \langle \theta, \dot{\gamma}(t) \rangle^2 dt \right)^2 \frac{d\theta}{2\pi} = \frac{L^2}{4}.$$

We claim that for every $\theta \in [0, 2\pi]$ the integrand above is

$$(G.3) \quad \int_0^L \langle \theta, \dot{\gamma}(t) \rangle^2 dt = \frac{L}{2},$$

which is certainly sufficient for (G.2).

To prove it we denote $\dot{\gamma}(t) =: e^{i\phi(t)}$ so that the integrand on the l.h.s. of (G.3) is

$$\langle \theta, \dot{\gamma}(t) \rangle^2 = \cos(\theta - \phi(t))^2,$$

and by the assumed invariance of \mathcal{C} w.r.t. rotations by $2\pi/k$ we have that

$$(G.4) \quad \int_0^L \cos(\theta - \phi(t))^2 dt = \int_0^{L/k} \left(\sum_{j=0}^{k-1} \cos \left(\theta - \phi(t) + j \cdot \frac{2\pi}{k} \right)^2 \right) dt.$$

Now, since, by assumption, $k \geq 3$, we have that

$$\sum_{j=0}^{k-1} \cos \left(\theta - \phi(t) + j \cdot \frac{2\pi}{k} \right)^2 = \frac{k}{2} + \sum_{j=0}^{k-1} \cos \left(2 \left(\theta - \phi(t) + \frac{2\pi}{k} \right) \right) = \frac{k}{2},$$

which we substitute into (G.4) to obtain

$$\int_0^L \cos(\theta - \phi(t))^2 dt = \frac{k}{2} \cdot \frac{L}{k} = \frac{L}{2},$$

that is, we obtain (G.3), which, as it was mentioned above, is sufficient to yield the statement of Proposition G.1.

□

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